A REGULAR SEQUENCE OF POLARIZATION OF MONOMIAL IDEAL

Let k be a field, and set $R = k[X_1, \ldots, X_d]$ and $\mathfrak{m} = \langle X_1, \ldots, X_d \rangle \leq R$. For each positive integer n, set $[n] = \{1, \ldots, n\}$. For each sequence of positive integers $\mathbf{n} = (n_1, \ldots, n_d)$ set $R_{\mathbf{n}} = k[X_{i,j} \mid i \in [d] \text{ and } j \in [n_i]]$.

Let I be a monomial ideal of R, and let $\widetilde{I} \leq R_{\mathbf{n}}$ be the polarization of I where $\mathbf{n}$ is determined by the generators of I. The first part of this project concludes that the sequence consisting of polynomials $X_{i,j-1} - X_{i,j}$ for all $i \in [d]$ and $1 < j \in [n_i]$ is regular for $R_{\mathbf{n}}/\widetilde{I}$. The goal of this part of the project is to show that the sequence consisting of polynomials $X_{i,1} - X_{i,j}$ for all $i \in [d]$ and $1 < j \in [n_i]$ is regular for $R_{\mathbf{n}}/\widetilde{I}$.

Exercise 1. Read and understand the statement and proof of [1, Theorem 16.5(ii)].

Exercise 2. Let A be a noetherian ring, let M be a finitely generated R-module, and let $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ be such that $(x_1, \ldots, x_n)A = (y_1, \ldots, y_n)A$ and there exists an invertible n-by-n matrix $Q := (q_{i,j})$ such that $[y_1 \ \cdots \ y_n]^T = Q\,[x_1 \ \cdots \ x_n]^T$. Assume that one of the following holds:

(α) $(A, \mathfrak{m})$ is a local ring and $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{m}$, or

(β) A is an $\mathbb{N}$-graded ring, M is an $\mathbb{N}$-graded A-module, and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are homogeneous of positive degree.

(a) Prove that $K^A(\underline{x}; M) \cong K^A(\underline{y}; M)$.

Proof. Since $K^A(\underline{x}; M) \cong K^A(\underline{x}) \otimes M$ and $K^A(\underline{y}; M) \cong K^A(\underline{y}) \otimes M$, it is equivalent to show that $K^A(\underline{x}) \cong K^A(\underline{y})$. Define $\psi_i : K^A(\underline{x})_i \to K^A(\underline{y})_i$ by $\psi_i = \bigwedge^i(Q^T)$ for $1 \leq i \leq n$ and $\psi_i = \mathrm{id}$, otherwise. Since $K^A(\underline{x})_j = 0 = K^A(\underline{y})_j$ for $j \geq n+1$ or for $j \leq -1$ and $\bigwedge^i(Q^T)$ is invertible for $1 \leq i \leq n$, to show $K^A(\underline{x}) \cong K^A(\underline{y})$, it suffices to show that the following diagram with $1 \leq i \leq n$ commutes.

$$\begin{array}{ccc}
K^A(\underline{y})_i & \xrightarrow{\partial_i^{K^A(\underline{y})}} & K^A(\underline{y})_{i-1} \\
\big\downarrow {\textstyle\bigwedge^i(Q^T)} & & \big\downarrow {\textstyle\bigwedge^{i-1}(Q^T)} \\
K^A(\underline{x})_i & \xrightarrow{\partial_i^{K^A(\underline{x})}} & K^A(\underline{x})_{i-1}
\end{array}$$

Let $\{e_1, \ldots, e_n\}$ be the standard A-basis of A^n. Then

$$\{e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i} \mid 1 \leq \lambda_1 < \cdots < \lambda_i \leq n\}$$

is an A-basis of $A^{\binom{n}{i}}$ for $2 \leq i \leq n$. Note that

$$\begin{aligned}
\partial_i^{K^A(\underline{y})}(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) &= \sum_{j=1}^{i} (-1)^{j-1} \partial_1^{K^A(\underline{y})}(e_{\lambda_j}) \cdot e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i} \\
&= \sum_{j=1}^{i} (-1)^{j-1} y_{\lambda_j} \cdot e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i},
\end{aligned}$$

and

$$\partial_i^{K^A(\underline{x})}(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) = \sum_{j=1}^{i} (-1)^{j-1} x_{\lambda_j} \cdot e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i}.$$

To show the above diagram commutes, it is enough to show that

$$\bigwedge^{i-1}(Q^T) \circ \partial_i^{K^A(\underline{y})}(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) = \partial_i^{K^A(\underline{x})} \circ \bigwedge^{i}(Q^T)(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}). \tag{2.1}$$

Claim. For $v_1 \wedge \cdots \wedge v_i \in K^A(\underline{y})_i$, we have

$$\partial_i^{K^A(\underline{y})}(v_1 \wedge \cdots \wedge v_i) = \sum_{j=1}^{i} (-1)^{j-1} \partial_1^{K^A(\underline{y})}(v_j) \cdot v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_i.$$

Proof by induction. Base case $i = 1$ follows from the definition. Inductive step. By [2, Exercise 4.33] and inductive hypothesis, we have

$$\begin{aligned}
&\partial_i^{K^A(\underline{y})}(v_1 \wedge \cdots \wedge v_i) \\
&= \partial_i^{K^A(\underline{y})}\big((v_1 \wedge \cdots \wedge v_{i-1}) \wedge v_i\big) \\
&= \partial_{i-1}^{K^A(\underline{y})}(v_1 \wedge \cdots \wedge v_{i-1}) \wedge v_i + (-1)^{i-1} \cdot v_1 \wedge \cdots \wedge v_{i-1} \cdot \partial_1^{K^A(\underline{y})}(v_i) \\
&= \left(\sum_{j=1}^{i-1} (-1)^{j-1} \partial_1^{K^A(\underline{y})}(v_j) \cdot v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{i-1} \right) \wedge v_i \\
&\quad + (-1)^{i-1} \cdot v_1 \wedge \cdots \wedge v_{i-1} \cdot \partial_1^{K^A(\underline{y})}(v_i) \\
&= \sum_{j=1}^{i-1} (-1)^{j-1} \partial_1^{K^A(\underline{y})}(v_j) \cdot v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{i-1} \wedge v_i \\
&\quad + (-1)^{i-1} \cdot \partial_1^{K^A(\underline{y})}(v_i) \cdot v_1 \wedge \cdots \wedge v_{i-1} \wedge \hat{v}_i \\
&= \sum_{j=1}^{i} (-1)^{j-1} \partial_1^{K^A(\underline{y})}(v_j) \cdot v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_i.
\end{aligned}$$

On the one hand, the left hand side of (2.1) is

$$\begin{aligned}
&\bigwedge^{i-1}(Q^T) \circ \partial_i^{K^A(\underline{y})}(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) \\
&= \bigwedge^{i-1}(Q^T) \left(\sum_{j=1}^{i} (-1)^{j-1} y_{\lambda_j} e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i} \right) \\
&= \sum_{j=1}^{i} (-1)^{j-1} y_{\lambda_j} \bigwedge^{i-1}(Q^T)(e_{\lambda_1} \wedge \cdots \wedge \hat{e}_{\lambda_j} \wedge \cdots \wedge e_{\lambda_i}) \\
&= \sum_{j=1}^{i} (-1)^{j-1} y_{\lambda_j} Q^T(e_{\lambda_1}) \wedge \cdots \wedge \hat{Q}^T(e_{\lambda_j}) \wedge \cdots \wedge Q^T(e_{\lambda_i}).
\end{aligned}$$

On the other hand, the right hand side of (2.1) is

$$\begin{aligned}
&\partial_i^{K^A(\underline{x})} \circ \bigwedge^i (Q^T)(e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_i}) \\
&= \partial_i^{K^A(\underline{x})} \big(Q^T(e_{\lambda_1}) \wedge \cdots \wedge Q^T(e_{\lambda_i})\big) \\
&= \sum_{j=1}^{i} (-1)^{j-1} \Big(\partial_1^{K^A}(\underline{x})\big(Q^T(e_{\lambda_j})\big)\Big) Q^T(e_{\lambda_1}) \wedge \cdots \wedge \hat{Q}^T(e_{\lambda_j}) \cdots \wedge Q^T(e_{\lambda_i}),
\end{aligned}$$

where

$$\begin{aligned}
\partial_1^{K^A}(\underline{x})\big(Q^T(e_{\lambda_j})\big) &= \partial_1^{K^A}(\underline{x})(q_{\lambda_j,1}e_1 + \cdots + q_{\lambda_j,n}e_n) \\
&= \partial_1^{K^A}(\underline{x})(q_{\lambda_j,1}e_1) + \cdots + \partial_1^{K^A}(\underline{x})(q_{\lambda_j,n}e_n) \\
&= q_{\lambda_j,1}x_1 + \cdots + q_{\lambda_j,n}x_n \\
&= y_{\lambda_j}.
\end{aligned}$$

Thus, (2.1) holds. □

(b) Use [1, Theorem 16.5(ii)] with part (a) to prove that $x_1, \ldots, x_n$ is M-regular if and only if $y_1, \ldots, y_n$ is M-regular.

Proof. $\Longrightarrow$ Assume $x_1, \ldots, x_n$ is M-regular. Then $\mathrm{H}_1(K^A(\underline{x}; M)) = \mathrm{H}_1(\underline{x}; M) = 0$ by [3, Theorem VIII.6.15]. Since $K^A(\underline{x}; M) \cong K^A(\underline{y}; M)$ by (a), we have

$$\mathrm{H}_1(\underline{y}; M) = \mathrm{H}_1(K^A(\underline{y}; M)) \cong \mathrm{H}_1(K^A(\underline{x}; M)) = 0.$$

So $y_1, \ldots, y_n$ is M-regular by [1, Theorem 16.5(ii)].
$\Longleftarrow$ It is similar to the proof of $\Longrightarrow$. □

Exercise 3. Use Exercise 2 to prove that the sequence consisting of polynomials $X_{i,1} - X_{i,j}$ for all $i \in [d]$ and $1 < j \in [n_i]$ is regular for $R_{\mathbf{n}}/\widetilde{I}$.

Proof. For $i = 1, \ldots, d$, define Q_{n_i} to be a lower triangular n_i-by-n_i matrix such that all the non-zero entries are 1. Then

$$\begin{bmatrix} X_{1,1} - X_{1,2} \\ \vdots \\ X_{1,1} - X_{1,i} \\ \vdots \\ X_{1,1} - X_{1,n_1} \\ \vdots \\ X_{d,1} - X_{d,2} \\ \vdots \\ X_{d,1} - X_{d,i} \\ \vdots \\ X_{d,1} - X_{d,n_d} \end{bmatrix} = \begin{bmatrix} Q_{n_1} & & \\ & \ddots & \\ & & Q_{n_d} \end{bmatrix} \begin{bmatrix} X_{1,1} - X_{1,2} \\ \vdots \\ X_{1,i-1} - X_{1,i} \\ \vdots \\ X_{1,n_1-1} - X_{1,n_1} \\ \vdots \\ X_{d,1} - X_{d,2} \\ \vdots \\ X_{d,i-1} - X_{d,i} \\ \vdots \\ X_{d,n_d-1} - X_{d,n_d} \end{bmatrix}$$

Since $Q := \begin{bmatrix} Q_{n_1} & & \\ & \ddots & \\ & & Q_{n_d} \end{bmatrix}$ is a lower triangular matrix whose diagonal entries are 1, we have Q is invertible. Since the first part of this project concludes that the sequence consisting of polynomials $X_{i,j-1} - X_{i,j}$ for all $i \in [d]$ and $1 < j < [n_i]$ is regular for $R_{\mathbf{n}}/\widehat{I}$, we have the sequence consisting of polynomials $X_{i,1} - X_{i,j}$ for all $i \in [d]$ and $1 < j \in [n_i]$ is also regular for $R_{\mathbf{n}}/\widetilde{I}$ by Exercise 2. $\square$

References

[1] H. Matsumura, *Commutative ring theory*, second ed., Studies in Advanced Mathematics, vol. 8, University Press, Cambridge, 1989. MR 90i:13001

[2] Kristen A Beck and Sean Sather-Wagstaff. A somewhat gentle introduction to differential graded commutative algebra. In *Connections Between Algebra, Combinatorics, and Geometry*, pages 3–99. Springer, 2014.

[3] Sean Sather-Wagstaff. Homological algebra notes. *Unpublished lecture notes. Available at* `http://ssather.people.clemson.edu/notes.html`, 2009.