COHEN-MACAULAY TYPE IN TERMS OF FREE RESOLUTION

Let k be a field, and set $R = k[X_1, \ldots, X_d]$ and $\mathfrak{X} = \langle X_1, \ldots, X_d \rangle \leq R$. Let I be an ideal of R generated by non-constant homogeneous polynomials. Assume that $\overline{R} = R/I$ is Cohen-Macaulay of dimension Δ.

Fact 1. There is a free resolution

$$F = (0 \to \underbrace{R^{\beta_{d-\Delta}}}_{\deg\ d-\Delta} \xrightarrow{\partial^F_{d-\Delta}} \cdots \xrightarrow{\partial^F_2} R^{\beta_1} \xrightarrow{\partial^F_1} R \to 0)$$

of $\overline{R}$ over R such that the entries of the matrices representing ∂^F_i are non-constant and homogeneous. Furthermore, one has $\operatorname{depth}_I(R) = d - \Delta$.

The goal of this project is to prove that $\beta_{d-\Delta} = \operatorname{type}(\overline{R})$. We accomplish this in steps.

Exercise 2. Let $\mathbf{f} = f_1, \ldots, f_\Delta \in \mathfrak{X}$ be a homogenous maximal $\overline{R}$-regular sequence. As in Homework 2, let $K = K^R(\mathbf{f}, F)$ be defined inductively as $K^R(f_\Delta, F) = \operatorname{Cone}(F \xrightarrow{f_\Delta} F)$ and $K = K^R(\mathbf{f}, F) = \operatorname{Cone}(K^R(\mathbf{f}', F) \xrightarrow{f_1} K^R(\mathbf{f}', F))$ where $\mathbf{f}' = f_2, \ldots, f_\Delta$.

(a) Prove that

$$K = K^R(\mathbf{f}, F) = (0 \to \underbrace{R^{\beta_{d-\Delta}}}_{\deg\ d} \xrightarrow{\partial^K_{d-\Delta}} \cdots \xrightarrow{\partial^K_2} R^{\Delta+\beta_1} \xrightarrow{\partial^K_1} R \to 0)$$

is a resolution of $\overline{R}/\langle \mathbf{f} \rangle \overline{R} \cong R/(I + \langle \mathbf{f} \rangle)$ over R such that the entries of the matrices representing ∂^K_i are non-constant and homogeneous.

Proof. Let F^+ be the corresponding augmented free resolution of $\overline{R}$:

$$F^+ = (0 \to \underbrace{R^{\beta_{d-\Delta}}}_{\deg\ d-\Delta} \xrightarrow{\partial^F_{d-\Delta}} \cdots \xrightarrow{\partial^F_2} R^{\beta_1} \xrightarrow{\partial^F_1} R \xrightarrow{\tau} \overline{R} \to 0).$$

It is enough to prove the claim: Let $\mathbf{g} = g_1, \ldots, g_D \in \mathfrak{X}$ be a homogeneous $\overline{R}$-regular sequence, then

$$L = K^R(\mathbf{g}, F) = (0 \to \underbrace{R^{\beta_{d-D}}}_{\deg\ d} \xrightarrow{\partial^L_{d-D}} \cdots \xrightarrow{\partial^L_2} R^{D+\beta_1} \xrightarrow{\partial^L_1} R \to 0)$$

is a resolution of $\overline{R}/\langle \mathbf{g} \rangle \overline{R} \cong R/(I + \langle \mathbf{g} \rangle)$ over R such that the entries of the matrices representing ∂^L_i are non-constant and homogeneous. To prove the claim, we use induction on D.

Base case: The case for $D = 0$ is covered in Fact 1. Let $D = 1$. Then

$$F^+ = (0 \to \underbrace{R^{\beta_{d-\Delta}}}_{\deg\ d-\Delta} \xrightarrow{\partial^F_{d-\Delta}} \cdots \xrightarrow{\partial^F_2} R^{\beta_1} \xrightarrow{\partial^F_1} R \xrightarrow{\tau} \overline{R} \to 0),$$

and $L = K^R(g_1, F) = \operatorname{Cone}\big(F \xrightarrow{g_1} F\big)$. So

$$\partial_1^L : \begin{matrix} F_0 \\ \oplus \\ F_1 \end{matrix} \xrightarrow{\begin{bmatrix} 0 & 0 \\ g_1 & \partial_1^F \end{bmatrix}} \begin{matrix} 0 \\ \oplus \\ F_0 \end{matrix} \Longrightarrow \partial_1^L : \begin{matrix} R \\ \oplus \\ R^{\beta_1} \end{matrix} \xrightarrow{\begin{bmatrix} g_1 & \partial_1^F \end{bmatrix}} \begin{matrix} 0 \\ \oplus \\ R \end{matrix} \Longrightarrow \partial_1^L : \ R^{1+\beta_1} \xrightarrow{\begin{bmatrix} g_1 & \partial_1^F \end{bmatrix}} R.$$

Since F^+ is exact, we have $\operatorname{Im}(\partial_1^F) = \operatorname{Ker}(\tau) = I$. So

$$\operatorname{Im}(\partial_1^L) = \operatorname{Im}\big(\begin{bmatrix} g_1 & \partial_1^F \end{bmatrix}\big) = \langle g_1 \rangle + \operatorname{Im}(\partial_1^F) = \langle g_1 \rangle + I.$$

Hence

$$\mathrm{H}_0(L) = R/\operatorname{Im}(\partial_1^L) = R/(\langle g_1 \rangle + I).$$

We claim that $\mathrm{H}_i(L) = 0$ for $i = 1, \ldots, d-1$. Since F is a resolution, we have $\mathrm{H}_i(F) = 0$ for $i \geq 1$ and $\mathrm{H}_0(F) \cong \overline{R}$. By Theorem I.D.20, the following sequence is exact:

$$0 \longrightarrow F \longrightarrow L \longrightarrow \Sigma F \longrightarrow 0.$$

We consider the long exact sequence of homology modules that rises from the above short exact sequence.

(1) Let $i \geq 2$. Then

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{H}_i(F) & \longrightarrow & \mathrm{H}_i(L) & \longrightarrow & \mathrm{H}_{i-1}(F) & \xrightarrow{g_1} & \mathrm{H}_{i-1}(F) & \longrightarrow \cdots \\ & \| & & & & \| & & \| & \\ & 0 & & & & 0 & & 0 & \end{array}$$

So by Fact I.B.2(c), we have $\mathrm{H}_i(L) = 0$.

(2) Let $i = 1$. Then

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{H}_1(F) & \longrightarrow & \mathrm{H}_1(L) & \longrightarrow & \mathrm{H}_0(F) & \xrightarrow{g_1} & \mathrm{H}_0(F) & \longrightarrow \cdots \\ & \| & & & & \downarrow\cong & & \downarrow\cong & \\ & 0 & & & & \overline{R} & \xrightarrow{g_1} & \overline{R} & \end{array}$$

Since $\mathrm{H}_1(F) = 0$ and the above sequence is exact, we have $\operatorname{Ker}\big(\mathrm{H}_1(L) \to \mathrm{H}_0(F)\big) = 0$. Since g_1 is a non-zero-divisor on $\overline{R}$, we have $\mathrm{H}_0(F) \xrightarrow{g_1} \mathrm{H}_0(F)$ is 1-1. So

$$\mathrm{H}_1(L) \cong \mathrm{H}_1(L)/\operatorname{Ker}(\mathrm{H}_1(L) \to \mathrm{H}_0(F)) \cong \operatorname{Ker}\big(\mathrm{H}_0(F) \xrightarrow{g_1} \mathrm{H}_0(F)\big) = 0.$$

Therefore, $\mathrm{H}_i(L) = 0$ for $i = 1, \ldots, d-1$. So $L = K^R(g_1, F)$ is a free resolution of $\mathrm{H}_0(L) = R/(\langle g_1 \rangle + I)$ by Lemma II.A.3. Since $g_1 \in \mathfrak{X}$ is homogenous and a non-zero-divisor on $\overline{R}$, we have g_1 is non-constant and homogeneous. Also, since the entries of the matrices representating ∂_i^F are non-constant and homogeneous, we have the entries of the matrices representating $\partial_i^L = \begin{bmatrix} -\partial_{i-1}^F & 0 \\ g_1 & \partial_i^F \end{bmatrix}$ are also non-constant and homogeneous.

Inductive case: Set $\mathbf{g}' = g_1, \ldots, g_{D-1}$ and $L' = K^R(\mathbf{g}', F)$. By definition, $\mathbf{g}'$ is $\overline{R}$-regular. The inductive hypothesis tells us that L' is a free resolution of $R/(\langle \mathbf{g}' \rangle + I)$ and the entries of the matrices representating $\partial_i^{L'}$ are non-constant and homogeneous. Then we claim that $(\langle \mathbf{g}' \rangle + I : g_D) = \langle \mathbf{g}' \rangle + I$.

Proof of claim. "$\supseteq$" follows from Proposition II.A.6. "$\subseteq$". Let $\alpha \in (\langle \mathbf{g}' \rangle + I : g_D)$, so $g_D \cdot \alpha \in \langle \mathbf{g}' \rangle + I$. Then $g_D \overline{\alpha} = \overline{g_D \alpha} = 0$ in $R/(\langle \mathbf{g}' \rangle + I)$. But g_D is a non-zero-divisor on $R/(\langle \mathbf{g}' \rangle + I) \cong \overline{R}/\langle \mathbf{g}' \rangle \overline{R}$ by condition (D) of Definition II.B.5, so $\overline{\alpha} = 0$ in $R/(\langle \mathbf{g}' \rangle + I)$. Therefore, $\alpha \in \langle \mathbf{g}' \rangle + I$.

Now consider the following free resolutions given by the inductive hypothesis:

$$\begin{array}{ccccccccccccc} & & & & & & & & & & R/(\langle \mathbf{g}' \rangle + I) & & \\ & & & & & & & & & & \| & & \\ (L')^+ = 0 & \longrightarrow & R^{\beta_{d-D+1}} & \longrightarrow & \cdots & \longrightarrow & R^{D-1+\beta_1} & \longrightarrow & R & \longrightarrow & R/(\langle \mathbf{g}' \rangle + I : g_D) & \longrightarrow & 0 \\ & & \downarrow g_D & & & & \downarrow g_D & & \downarrow g_D & & \hookdownarrow g_D & & \\ (L')^+ = 0 & \longrightarrow & R^{\beta_{d-D+1}} & \longrightarrow & \cdots & \longrightarrow & R^{D-1+\beta_1} & \longrightarrow & R & \longrightarrow & R/(\langle \mathbf{g}' \rangle + I) & \longrightarrow & 0 \end{array}$$

By Theorem II.A.7,

$$L = K^R(\mathbf{g}, F) = \operatorname{Cone}\big(K^R(\mathbf{g}', F) \xrightarrow{g_D} K^R(\mathbf{g}', F)\big) = \operatorname{Cone}\big(L' \xrightarrow{g_D} L'\big)$$

is a free resolution of $R/(\langle \mathbf{g}' \rangle + I + g_D R) = R/(\langle \mathbf{g} \rangle + I)$. Since $\partial_i^L = \begin{bmatrix} -\partial_{i-1}^{L'} & 0 \\ g_1 & \partial_i^{L'} \end{bmatrix}$ and the entries of the matrices representing $\partial_i^{L'}$ are non-constant and homogeneous and g_1 is non-constant and homogeneous, we have the entries of the matrices representing ∂_i^L are non-constant and homogeneous. □

(b) Since type $(\overline{R})$ = type $(\overline{R}/\langle \mathbf{f} \rangle)$, conclude that we may assume without loss of generality that $\Delta = 0$.

Proof. We need to show that $\beta_{d-\Delta} = \operatorname{type}(\overline{R})$, it is enough to show that $\beta_{d-\Delta} = \operatorname{type}(\overline{R}/\langle \mathbf{f} \rangle)$ since $\operatorname{type}(\overline{R}) = \operatorname{type}(\overline{R}/\langle \mathbf{f} \rangle)$. But part (a) gives a free resolution for $\overline{R}/\langle \mathbf{f} \rangle$, which is Cohen-Macaulay of dimension $\dim(\overline{R}) - \Delta = \Delta - \Delta = 0$, so we may assume without losss of generality that $\Delta = 0$. □

Remark. We can also just use Theorem II.A.7 to prove the base case $D = 1$ in (a).

Assume for the rest of the project that $\Delta = 0$. It follows that we have $\operatorname{type}(\overline{R}) = \dim_k(\operatorname{Hom}_R(k, \overline{R})) = \dim_k(\operatorname{Hom}_{\overline{R}}(k, \overline{R}))$, and the goal is to prove that $\beta_d = \operatorname{type}(\overline{R})$.

Exercise 3. (a) Use Fact 1 to prove that $\operatorname{Ext}_R^i(\overline{R}, R) = 0$ for all $i \neq d$.

Proof. We have

$$F^* : 0 \to \operatorname{Hom}_R(R, R) \xrightarrow{(\partial_1^F)^*} \cdots \xrightarrow{(\partial_d^F)^*} \operatorname{Hom}_R(R^{\beta_d}, R) \to 0.$$

Since $(F^*)_j = F_{-j}^* = 0^* = 0$ for all $j \leq -d-1$, we have

$$\operatorname{Ext}_R^i(\overline{R}, R) = \frac{\operatorname{Ker}(\partial_{-i}^{F^*})}{\operatorname{Im}(\partial_{-i+1}^{F^*})} = \frac{\operatorname{Ker}(0 \to (F^*)_{-i-1})}{\operatorname{Im}(\partial_{-i+1}^{F^*})} = 0, \ \forall\, i \geq d+1.$$

Since $\Delta = 0$, we have $\operatorname{depth}_I(R) = d - \Delta = d$. So there exists a R-regular sequence in I of length d, which is also weakly R-regular. So $\operatorname{Ext}_R^i(\overline{R}, R) = 0$ for all $i \leq d-1$ by Theorem II.C.4(a). □

(b) Prove that $\Sigma^d F^* = \Sigma^d \operatorname{Hom}_R(F, R)$ is a free resolution of $\omega := \operatorname{Ext}_R^d(\overline{R}, R)$.

Proof. We have

$$\Sigma^d F^* = (0 \to \underbrace{\operatorname{Hom}_R(R, R)}_{\deg d} \xrightarrow{(-1)^d(\partial_1^F)^*} \cdots \xrightarrow{(-1)^d(\partial_d^F)^*} \operatorname{Hom}_R(R^{\beta_d}, R) \to 0),$$

implying

$$\Sigma^d F^* = (0 \to R \xrightarrow{(-1)^d(\partial_1^F)^*} \cdots \xrightarrow{(-1)^d(\partial_d^F)^*} R^{\beta_d} \to 0).$$

By (a) we have $\mathrm{H}_j(F^*) = \mathrm{Ext}_R^{-j}(\overline{R}, R) = 0$ for $j \geq 1 - d$. Then by Remark I.D.7, we have $\mathrm{H}_i(\Sigma^d F^*) = \mathrm{H}_{i-d}(F^*) = 0$ for $i \geq 1$. Also note that $(\Sigma^d F^*)_i = (F^*)_{i-d}$ is free for each i. So by Lemma II.A.3, we have $\Sigma^d F^*$ is a free resolution of $\mathrm{H}_0(\Sigma^d F^*) \cong \mathrm{H}_{-d}(F^*) = \mathrm{Ext}_R^d(\overline{R}, R) = \omega$. □

(c) Use Nakayama's lemma to prove that ω is minimally generated by β_d many elements.

Proof. Note that

$$\omega = \mathrm{Ext}_R^d(\overline{R}, R) = \frac{\mathrm{Ker}(\partial_{-d}^{F^*})}{\mathrm{Im}(\partial_{-d+1}^{F^*})} = \frac{\mathrm{Ker}((F_d)^* \to 0)}{\mathrm{Im}((\partial_d^F)^*)} \cong \frac{R^{\beta_d}}{\mathrm{Im}((\partial_d^F)^*)}.$$

Let $C \in \mathrm{Mat}_{\beta_{d-1} \times \beta_d}(R)$ be the matrix representing $\partial_d^F : R^{\beta_d} \to R^{\beta_{d-1}}$. Then $D := C^T$ is the matrix representing $(\partial_d^F)^* : R^{\beta_{d-1}} \to R^{\beta_d}$. Since the entries $C_{i,j}$ are non-constant and homogeneous, we have $C_{i,j} \in \mathfrak{X}$ and then $D_{j,i} \in \mathfrak{X}$. Hence we have $\mathrm{Im}((\partial_d^F)^*) = D(R^{\beta_{d-1}}) \subseteq \mathfrak{X} R^{\beta_d}$ is a submodule. So by the third isomorphism theorem for modules,

$$\frac{\omega}{\mathfrak{X}\omega} \cong \frac{R^{\beta_d}}{\mathfrak{X} R^{\beta_d} + \mathrm{Im}((\partial_d^F)^*)} = \frac{R^{\beta_d}}{\mathfrak{X} R^{\beta_d}} = \frac{R^{\beta_d}}{(\mathfrak{X})^{\beta_d}} \cong \left(\frac{R}{\mathfrak{X}}\right)^{\beta_d} \cong k^{\beta_d}.$$

Since the length of basis of the k-vector space k^{β_d} is β_d, we have the length of basis of the k-vector space $\frac{\omega}{\mathfrak{X}\omega}$ is β_d. So the R-module ω is minimally generated by β_d many elements by Nakayama's lemma. □

Fact 4. Let M be an R-module. Then M has a well-defined $\overline{R}$-module structure defined by the formula $\bar{r}m := rm$ if and only if $IM = 0$.

Fact 5. If M is an $\overline{R}$-module and N is an R-module, then $\mathrm{Ext}_R^i(M, N)$ and $\mathrm{Ext}_R^i(N, M)$ are $\overline{R}$-modules for $i \in \mathbb{Z}$.

Proof. Since $\mathrm{Ext}_R^i(M, N)$ and $\mathrm{Ext}_R^i(N, M)$ are R-modules, by Fact 4 it suffices to show that $I \,\mathrm{Ext}_R^i(M, N) = 0 = I \,\mathrm{Ext}_R^i(N, M)$. Since M is an $\overline{R}$-module, $IM = 0$ by Fact 4. Then $\mu^{M,a} : M \xrightarrow{a} M$ is the zero map for all $a \in I$. So for all $a \in I$:

$$\begin{aligned} \mu^{\mathrm{Ext}_R^i(M,N),a} &= \mathrm{Ext}_R^i(\mu^{M,a}, N) = \mathrm{Ext}_R^i(0, N) = 0 \\ &= \mathrm{Ext}_R^i(N, 0) = \mathrm{Ext}_R^i(N, \mu^{M,a}) = \mu^{\mathrm{Ext}_R^i(N,M),a}. \end{aligned}$$

Hence

$$a \cdot \mathrm{Ext}_R^i(M, N) = 0 = a \cdot \mathrm{Ext}_R^i(N, M), \ \forall\, a \in I.$$

Thus, $I \,\mathrm{Ext}_R^i(M, N) = 0 = I \,\mathrm{Ext}_R^i(N, M)$. □

Fact 6. Let M, N be $\overline{R}$-modules and $f : M \to N$ a function. Then f is an R-module homomorphism if and only if it is an $\overline{R}$-module homomorphism. In other words, $\mathrm{Hom}_{\overline{R}}(M, N) = \mathrm{Hom}_R(M, N)$.

Proof. Let $\bar{r} \in \overline{R}$ with $r \in R$ and $m \in M$. Since M, N are $\overline{R}$-modules, we have $f(\bar{r}m) = f(rm)$ and $\bar{r}f(m) = rf(m)$. So $f(\bar{r}m) = \bar{r}f(m)$ if and only if $f(rm) = rf(m)$. □

Fact 7. Given R-complexes A, B, C one can construct Hom-complexes and tensor-product-complexes such that there is an isomorphism

$$\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)) \cong \mathrm{Hom}_R(A \otimes_R B, C). \tag{7.1}$$

Also, if P is free resolution of M, and Q is a free resolution of N, then for all i we have

$$\mathrm{H}_{-i}(\mathrm{Hom}_R(P,Q)) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P,N)) \cong \mathrm{Ext}^i_R(M,N) \tag{7.2}$$

$$\mathrm{Hom}_R(\Sigma^i A, \Sigma^i B) \cong \mathrm{Hom}_R(A,B) \tag{7.3}$$

$$\mathrm{H}_{-i}(P \otimes_R Q) \cong \mathrm{H}_{-i}(P \otimes_R N) \tag{7.4}$$

$$A \otimes_R B \cong B \otimes_R A \tag{7.5}$$

$$(\Sigma^i A) \otimes_R (\Sigma^{-i} B) \cong A \otimes_R B \tag{7.6}$$

In particular, one has $\mathrm{Hom}_{\overline{R}}(\omega,\omega) \cong \overline{R}$ because ω is an $\overline{R}$-module by Fact 5 and

$$\begin{aligned}
\mathrm{Hom}_{\overline{R}}(\omega,\omega) &= \mathrm{Hom}_R(\omega,\omega) && \text{by Fact 6}\\
&\cong \mathrm{Ext}^0_R(\omega,\omega)\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(\Sigma^d F^*, \Sigma^d F^*)) && \text{by (7.2)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F^*, F^*)) && \text{by (7.3)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F^*, \mathrm{Hom}_R(F,R)))\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F^* \otimes_R F, R)) && \text{by (7.1)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F \otimes_R F^*, R)) && \text{by (7.5)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F, \mathrm{Hom}_R(F^*, R)))\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F, F^{**}))\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F, F))\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F, \overline{R})) && \text{by (7.2)}\\
&\cong \mathrm{Ext}^0_R(\overline{R}, \overline{R})\\
&\cong \mathrm{Hom}_R(\overline{R}, \overline{R})\\
&\cong \overline{R}
\end{aligned}$$

Exercise 8. Let L be a complex of finite rank free R-modules, and let N be an R-module. Prove that the natural map $\Phi\colon L^* \otimes_R N \to \mathrm{Hom}_R(L,N)$ given by $\Phi(\alpha \otimes n)(x) = \alpha(x)n$ is an isomorphism of R-complexes. (Hint: Prove that it is a chain map, then prove that it is an isomorphism when $L = R^b$.)

Proof. Let $i \in \mathbb{Z}$. Define

$$\begin{aligned}
\phi : L^*_{-i} \times N &\longrightarrow \mathrm{Hom}_R(L_{-i}, N)\\
\phi(\alpha, n)(x) &\longmapsto \alpha(x)n.
\end{aligned}$$

Then to prove Φ_i is a well-defined R-module homomorphism, we need to show that ϕ is a well-defined R-bilinear function. Let $(\alpha, n) \in L^*_{-i} \times N$. Then $\alpha \in L^*_{-i} = \mathrm{Hom}_R(L_{-i}, R)$. Let $l_1, l_2 \in L_{-i+1}$ and $r \in R$. Then

$$\begin{aligned}
\phi(\alpha,n)(rl_1 + l_2) = \alpha(rl_1 + l_2)n = \big(r\alpha(l_1) + \alpha(l_2)\big)n &= r\alpha(l_1)n + \alpha(l_2)n\\
&= r\phi(\alpha,n)(l_1) + \phi(\alpha,n)(l_2).
\end{aligned}$$

So $\phi(\alpha,n) \in \mathrm{Hom}_R(L_{-i}, N)$. Hence ϕ is well-defined. Let $\alpha_1, \alpha_2, \alpha \in L^*_{-i}$, $n_1, n_2, n \in N$ and $r, s \in R$. Then for $x \in L_{-i+1}$ we have

$$\phi(r\alpha_1 + \alpha_2, n)(x) = (r\alpha_1 + \alpha_2)(x)n = r\alpha_1(x)n + \alpha_2(x)n = r\phi(\alpha_1,n)(x) + \phi(\alpha_2,n)(x),$$

$$\phi(\alpha, n_1 s + n_2)(x) = \alpha(x)(n_1 s + n_2) = (\alpha(x)n_1)s + \alpha(x)n_2 = (\phi(\alpha,n_1)s)(x) + \phi(\alpha,n_2)(x).$$

So $\phi(r\alpha_1+\alpha_2,n)=r\phi(\alpha_1,n)+\phi(\alpha_2,n)$ and $\phi(\alpha,n_1s+n_2)=\phi(\alpha,n_1)s+\phi(\alpha,n_2)$. Hence ϕ is R-bilinear. Consider the following diagram.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & L_i^*\otimes_R N & \xrightarrow{\partial_i^{L^*}\otimes_R N} & L_{i-1}^*\otimes_R N & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \Phi_i} & & \downarrow{\scriptstyle \Phi_{i-1}} & & \\
\cdots & \longrightarrow & \mathrm{Hom}_R(L_{-i},N) & \xrightarrow{\mathrm{Hom}_R(\partial_{-i+1}^{L},N)} & \mathrm{Hom}_R(L_{-i+1},N) & \longrightarrow & \cdots
\end{array}$$

To show the commutativity of the above diagram, it is enough to show that it is commutative on the generators of $L_i^*\otimes_R N$. Let $\alpha\otimes n\in L_i^*\otimes_R N$. Then for $x\in L_{-i+1}$, we have

$$\begin{aligned}
\Phi_{i-1}((\partial_i^{L^*}\otimes_R N)(\alpha\otimes n))(x) &= \Phi_{i-1}(\partial_i^{L^*}(\alpha)\otimes n)(x)=(\partial_i^{L^*}(\alpha))(x)n\\
&=((\partial_{-i+1}^{L})^*(\alpha))(x)n=(\alpha\circ\partial_{-i+1}^{L})(x)n,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Hom}_R(\partial_{-i+1}^{L},N)(\Phi_i(\alpha\otimes n))(x) &= (\Phi_i(\alpha\otimes n)\circ\partial_{-i+1}^{L})(x)=\Phi_i(\alpha\otimes n)(\partial_{-i+1}^{L}(x))\\
&=\alpha(\partial_{-i+1}^{L}(x))n=(\alpha\circ\partial_{-i+1}^{L})(x)n.
\end{aligned}$$

So $\Phi_{i-1}((\partial_i^{L^*}\otimes_R N)(\alpha\otimes n))=\mathrm{Hom}_R(\partial_{-i+1}^{L},N)(\Phi_i(\alpha\otimes n))$ and thus $\Phi_{i-1}\circ(\partial_i^{L^*}\otimes_R N)=\mathrm{Hom}_R(\partial_{-i+1}^{L},N)\circ\Phi_i$. Hence Φ is a chain map. Assume without loss of generality that $L_j=R^{b_j}$ for all $j\in\mathbb{Z}$. Then $L_i^*=(L_{-i})^*=\mathrm{Hom}_R(L_{-i},R)=\mathrm{Hom}_R(R^{b_{-i}},R)$. To prove Φ is an isomorphism, it is enough to show that Φ_i is bijective. Let $\{e_\lambda\}_{\lambda=1}^{b_{-i}}\subseteq R^{b_{-i}}$ be a basis. Then we have the following isomorphisms

$$\begin{aligned}
\mathrm{Hom}_R(R^{b_{-i}},R) &\xrightarrow{\cong} R^{b_{-i}}\\
\psi &\longmapsto (\psi(e_\lambda)),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Hom}_R(R^{b_{-i}},N) &\xrightarrow{\cong} N^{b_{-i}}\\
\phi &\longmapsto (\phi(e_\lambda)),
\end{aligned}$$

and

$$\begin{aligned}
R^{b_{-i}}\otimes_R N &\longrightarrow N^{b_{-i}}\\
(e_\lambda)\otimes n &\longmapsto (e_\lambda n).
\end{aligned}$$

So we have an induced isomorphism:

$$\begin{aligned}
\mathrm{Hom}_R(R^{b_{-i}},R)\otimes_R N &\xrightarrow{\cong} R^{b_{-i}}\otimes_R N \xrightarrow{\cong} N^{b_{-i}}\\
\alpha\otimes n &\longmapsto (\alpha(e_\lambda))\otimes n\longmapsto(\alpha(e_\lambda)n).
\end{aligned}$$

Hence the following diagram commutes.

$$\begin{array}{ccccc}
\alpha\otimes n & & \longrightarrow & & \alpha(\cdot)n\\
\Big\downarrow & \mathrm{Hom}_R(R^{b_{-i}},R)\otimes_R N & \xrightarrow{\Phi_i} & \mathrm{Hom}_R(R^{b_{-i}},N) & \Big\downarrow\\
 & {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} & \\
 & N^{b_{-i}} & \xrightarrow{=} & N^{b_{-i}} & \\
(\alpha(e_\lambda)n) & & \longrightarrow & & (\alpha(e_\lambda)n)
\end{array}$$

Thus, Φ_i is an isomorphism. □

Exercise 9. (a) Use the Koszul resolution K of k over R to prove that

$$\mathrm{Hom}_{\overline{R}}(k,\omega) = \mathrm{Hom}_R(k,\omega) \cong \mathrm{Ext}^0_R(k,\omega) \cong \mathrm{H}_0(F^* \otimes_R k) \cong k.$$

Proof. Since $I \subseteq \mathfrak{X}$, we have $Ik = 0$. So k is an $\overline{R}$-module. Also, since ω is an $\overline{R}$-module by Fact 5, we have $\mathrm{Hom}_{\overline{R}}(k,\omega) = \mathrm{Hom}_R(k,\omega)$ by Fact 6. The first isomorphism is by the proof of Theorem I.E.6. Since K is a free resolution of k and $\Sigma^d F^*$ is a free resolution of ω by Exercise 3(b),

$$\begin{aligned}
\mathrm{Ext}^0_R(k,\omega) &\cong \mathrm{H}_0(\mathrm{Hom}_R(K,\Sigma^d F^*)) && \text{by (7.2)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(\Sigma^d K^*,\Sigma^d F^*)) && K \text{ is self-dual}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(K^*,F^*)) && \text{by (7.3)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(K^*,\mathrm{Hom}_R(F,R))) &&\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(K^*\otimes_R F,R)) && \text{by (7.1)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F\otimes_R K^*,R)) && \text{by (7.5)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F,\mathrm{Hom}_R(K^*,R))) && \text{by (7.1)}\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F,K^{**})) &&\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F,K)) &&\\
&\cong \mathrm{H}_0(\mathrm{Hom}_R(F,k)) && \text{by (7.2)}\\
&\cong \mathrm{H}_0(F^*\otimes_R k) && \text{by Exercise 8}
\end{aligned}$$

Note that

$$F^* \otimes_R k : 0 \to R \otimes_R k \xrightarrow{(\partial_1^F)^*\otimes k} \cdots \xrightarrow{(\partial_d^F)^*\otimes k} R^{\beta_d} \otimes_R k \to 0,$$

implying

$$F^*/\mathfrak{X}F^* \cong F^* \otimes_R k : 0 \to k \xrightarrow{(\partial_1^F)^*\otimes R/\mathfrak{X}} k^{\beta_1} \to \cdots \xrightarrow{(\partial_d^F)^*\otimes R/\mathfrak{X}} k^{\beta_d} \to 0.$$

Note that

$$\mathrm{Im}\big((\partial_1^F)^* \otimes R/\mathfrak{X}\big) = \mathrm{Im}((\partial_1^F)^*) \otimes_R R/\mathfrak{X} \cong \mathrm{Im}((\partial_1^F)^*)/\mathfrak{X}\,\mathrm{Im}((\partial_1^F)^*).$$

Similar to Exercise 3(c), we have $\mathrm{Im}((\partial_1^F)^*) \subseteq \mathfrak{X}$. So $\mathrm{Im}\big((\partial_1^F)^* \otimes R/\mathfrak{X}\big) = 0$. Hence

$$\mathrm{H}_0(F^* \otimes_R k) \cong \mathrm{Ker}\big((\partial_1^F)^* \otimes R/\mathfrak{X}\big) = k.$$

□

(b) Prove that $\mathrm{Hom}_{\overline{R}}(k,\overline{R}) \cong \mathrm{Hom}_{\overline{R}}(k,\omega)^{\beta_d} \cong k^{\beta_d}$.

Proof. By (a), it is enough to show that $\mathrm{Hom}_{\overline{R}}(k,\overline{R}) \cong \mathrm{Hom}_{\overline{R}}(k,\omega)^{\beta_d}$. Since ω is minimally generated β_d many elements by Exercise 3(c), we have $k^{\beta_d} \cong \omega \otimes_R k \cong k \otimes_R \omega$ by Lemma VII.3.12 in Homological Algebra Notes and by (7.5). So

$$\begin{aligned}
\mathrm{Hom}_{\overline{R}}(k,\omega)^{\beta_d} &\cong \mathrm{Hom}_{\overline{R}}(k^{\beta_d},\omega)\\
&\cong \mathrm{Hom}_{\overline{R}}(k\otimes_R \omega,\omega)\\
&\cong \mathrm{Hom}_{\overline{R}}(k,\mathrm{Hom}_{\overline{R}}(\omega,\omega)) && \text{by (7.1)}\\
&\cong \mathrm{Hom}_{\overline{R}}(k,\overline{R}) && \text{by Fact 7.}
\end{aligned}$$

□

(c) Conclude that $\mathrm{type}(\overline{R}) = \beta_d$, as desired.

Proof. By Fact 6 and proof of (a), we have $\operatorname{Hom}_R(k, \overline{R}) = \operatorname{Hom}_{\overline{R}}(k, \overline{R})$. Since $\operatorname{depth}(\overline{R}) \leq \dim(\overline{R}) = \Delta = 0$, we have $\operatorname{depth}(\overline{R}) = 0$. So by (b),

$$\operatorname{type}(\overline{R}) = \dim_k(\operatorname{Ext}^0_R(k, \overline{R})) = \dim_k(\operatorname{Hom}_R(k, \overline{R})) = \dim_k(\operatorname{Hom}_{\overline{R}}(k, \overline{R})) = \beta_d. \qquad \square$$