Contents

Chapter 1

Fundamental Properties of Monomial Ideals

Let A be a nonzero commutative ring with identity and $R = A[X_1, \dots, X_d]$.

1.1 Monomial Ideals

Notation 1.1.

$$(n_1, \dots, n_d) = \underline{n} \in \mathbb{N}_0^d.$$

$$\underline{X} = X_1^{n_1} \cdots X_d^{n_d},$$

which is *monomials* in $X_1, \dots, X_d$.

Remark. A monomial is nonzero.

Definition 1.2. A *monomial ideal* I in R is an ideal generated by a set of monomials in I or R. Denote this as $I \leqslant_m R$.

Definition 1.3. If $I \leqslant_m R$, then

$$[\![I]\!] = \{\text{monomials in } I\} = I \cap [\![A[X_1, \dots, X_d]]\!] = I \cap [\![R]\!].$$

Lemma 1.4. Let $I \leqslant_m R$, then $(I \cap [\![R]\!]) = ([\![I]\!]) = I$.

Proof. Let $I = (S)$, where $S \subseteq [\![I]\!]$. Then $I = (S) \subseteq ([\![I]\!]) \subseteq I$. □

Theorem 1.5. *Let $I, J \leqslant_m R$. Then*

(a) $I \subseteq J$ if and only if $[\![I]\!] \subseteq [\![J]\!]$.

(b) $I = J$ if and only if $[\![I]\!] = [\![J]\!]$.

Proof. (a) It folllows from $I = ([\![I]\!])$ and $J = ([\![J]\!])$.

(b) By (a), $I = J$ if and only if $I \subseteq J$ and $J \subseteq I$ if and only if $[\![I]\!] \subseteq [\![J]\!]$ and $[\![J]\!] \subseteq [\![I]\!]$. □

Definition 1.6. The d-tuple $\underline{n} \in \mathbb{N}_0^d$ is the *exponent vector* of $\underline{X}^{\underline{n}} =: f \in [\![R]\!]$.

Definition 1.7. Let $f, g \in [\![R]\!]$. f is *monomial multiple* of g if $f = gh$ for some $h \in [\![R]\!]$.

Remark (Notation). Let $\underline{m}, \underline{n} \in \mathbb{N}_0^d$. $\underline{m} \succcurlyeq \underline{n}$ means $m_i \geqslant n_i$ for $i = 1, \dots, d$.

Lemma 1.8. Let $f = \underline{X}^{\underline{m}}, g = \underline{X}^{\underline{n}} \in [\![R]\!]$ and $h \in R$. If $f = gh$, then $h \in [\![R]\!]$, $\underline{m} \succcurlyeq \underline{n}$ and $h = \underline{X}^{\underline{p}}$, where $\underline{p} = \underline{m} - \underline{n}$.

Proof. Let $h = \sum_{\underline{p} \in \Lambda} \alpha_{\underline{p}} \underline{X}^{\underline{p}}$, where $\Lambda \subseteq \mathbb{N}_0^d$ is finite and $0 \neq \alpha_{\underline{p}} \in A$ for $\underline{p} \in \Lambda$. Then $\underline{X}^{\underline{m}} = f = gh = \underline{X}^{\underline{n}} \sum_{\underline{p} \in \Lambda} \alpha_{\underline{p}} \underline{X}^{\underline{p}} = \sum_{\underline{p} \in \Lambda} \alpha_{\underline{p}} \underline{X}^{\underline{n}+\underline{p}}$. Since monomials in R are A-linear independent as A-module, $\alpha_{\underline{p}} = \begin{cases} 1 & \text{if } \underline{n} + \underline{p} = \underline{m} \\ 0 & \text{if } \underline{n} + \underline{p} \neq \underline{m} \end{cases}$ for $\underline{p} \in \Lambda$. So $gh = f = \underline{X}^{\underline{m}} = \underline{X}^{\underline{n}} \underline{X}^{\underline{p}} = g\underline{X}^{\underline{p}}$. Since $\underline{X}^{\underline{p}} \in \mathrm{NZD}(R)$, $h = \underline{X}^{\underline{p}}$. Also, $\underline{m} = \underline{n} + \underline{p} \succcurlyeq \underline{n}$ and $\underline{p} = \underline{m} - \underline{n}$. □

Lemma 1.9. Let $R = A[X_1, \dots, X_n]$ and $f = \underline{X}^{\underline{m}}$ and $\underline{g} = \underline{X}^{\underline{n}}$. The followings are equivalent.

(a) $f \in gR$.

(b) f is a multiple of g.

(c) f is a monomial multiple of g.

(d) $\underline{m} \succcurlyeq \underline{n}$.

(e) $\underline{m} \in \langle \underline{n} \rangle := \{\underline{p} \in \mathbb{N}_0^d | \underline{p} \succcurlyeq \underline{n}\}$.

Proof. (d)$\Longrightarrow$(c) If $\underline{m} \succcurlyeq \underline{n}$, then $\underline{p} := \underline{m} - \underline{n} \in \mathbb{N}^d$. Let $h = \underline{X}^{\underline{p}}$, then $gh = X^{\underline{n}+\underline{p}} = \underline{X}^{\underline{m}} = f$. So f is a monomial multiple of g. □

Theorem 1.10. *Let $f, f_1, \dots, f_n \in [\![R]\!]$. Then $f \in \langle f_1, \dots, f_n \rangle$ if and only if $f \in \langle f_i \rangle$ for some $i \in \{1, \dots, n\}$.*

Proof. $\Longleftarrow$ It is straightforward.

$\Longrightarrow$ Let $f \in \langle f_1, \dots, f_n \rangle$. Assume $f = \underline{X}^{\underline{n}}$ and $f_i = \underline{X}^{\underline{n_i}}$ for $i = 1, \dots, n$. Then $f = \sum_{i=1}^n g_i f_i$, where $g_i = \sum_{\underline{p}}^{\text{finite}} \alpha_{i,\underline{p}} \underline{X}^{\underline{p}} \in R$. So $\underline{X}^{\underline{n}} = f = \sum_{i=1}^n g_i f_i = \sum_{i=1}^n \sum_{\underline{p}}^{\text{finite}} \alpha_{i,\underline{p}} \underline{X}^{\underline{n_i}+\underline{p}}$. Hence there exists $i \in \{1, \dots, n\}$ and $\underline{p} \in \mathbb{N}_0^d$ such that $f = \underline{X}^{\underline{n}} = \underline{X}^{\underline{n_i}+\underline{p}} = f_i \underline{X}^{\underline{p}}$. □

Lemma 1.11. Let $I \leqslant R$. The followings are equivalent.

(a) $I \leqslant_m R$.

(b) For $f \in I$, each monomial occuring in f is in I.

Example 1.12. If $I \leqslant_m A[X, Y, Z]$ and $X^2 + XZ + YZ \in I$, then $X^2, XZ, YZ \in I$.

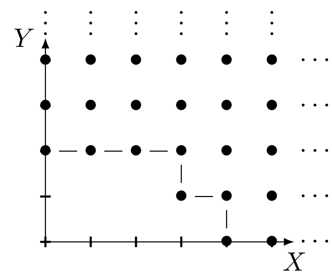
Definition 1.13. Let $I \leqslant_m R$. The *graph* of I is

$$\Gamma(I) = \{\underline{n} \in \mathbb{N}_0^d \mid \underline{X}^{\underline{n}} \in I\} = \{\text{exponent vector for } f \in [\![I]\!]\} \subseteq \mathbb{N}_0^d.$$

Theorem 1.14. *If $I = \langle \underline{X}^{\underline{n_1}}, \dots, \underline{X}^{\underline{n_t}} \rangle$, then $\Gamma(I) = \langle \underline{n_1} \rangle \cup \dots \cup \langle \underline{n_t} \rangle$.*

Proof. $\underline{m} \in \Gamma(I)$ if and only if $\underline{X}^{\underline{m}} \in I$ if and only if $\underline{X}^{\underline{m}} \in \langle \underline{X}^{\underline{n_1}}, \ldots, \underline{X}^{\underline{n_t}} \rangle$ if and only if $\underline{X}^{\underline{m}} \in \langle \underline{X}^{\underline{n_i}} \rangle$ for some $i \in \{1, \ldots, t\}$ if and only if $\underline{m} \succcurlyeq \underline{n_i}$ for some $i \in \{1, \ldots, t\}$ if and only if $\underline{m} \in \langle \underline{n_i} \rangle$ for some $i \in \{1, \ldots, t\}$ if and only if $\underline{m} \in \bigcup_{i=1}^{t} \langle \underline{n_i} \rangle$. □

Example 1.15. Let $I = \langle X^4, X^3Y, Y^2 \rangle \leqslant_m A[X,Y]$. Then $\Gamma(I) = \langle (4,0) \rangle \cup \langle (3,1) \rangle \cup \langle (0,2) \rangle \subseteq \mathbb{N}^2$.



1.2 Generators of monomial ideals

Remark (Facts)**.** Let R be a commutative ring with identity, $S \subseteq R$ and $I = (S)$. If I is finitely generated over R, then there exist $s_1, \ldots, s_n \in S$ such that $I = (s_1, \ldots, s_n)$.

Theorem 1.16 (Dickson's lemma)**.** *Let $I \leqslant_m R$. Then I is finitely generated by a list of monomials.*

Proof. Induct on $d \geqslant 1$. Base case: $d = 1$. $I \leqslant_m R = A[X]$. If $I = 0$, then $I = (\emptyset)$. Assume $I \neq 0$. Let $r = \min\{n \geqslant 0 \mid X^n \in I\}$. Then $r < \infty$. So $X^r \in I$ and then $(X^r) \subseteq I$. Let $X^s \in [\![I]\!]$. Then $r \leqslant s$ by the minimality of r. So $X^s \in (X^r)$. Hence $[\![I]\!] \subseteq (X^r)$. Since $I = ([\![I]\!])$ is the smallest ideal containing $[\![I]\!]$, $I \subseteq (X^r)$.

Inductive step. $d \geqslant 2$. Let $R' = A[X_1, \ldots, X_{d-1}] \subseteq R$. Assume the statement is true for R'. Let $I \leqslant_m R$. Set $S = \{z \in [\![R']\!] \mid zX_d^a \in I \text{ for some } a \in \mathbb{N}_0\}$ and $J = (S)R'$. Then $J \leqslant_m R'$. By inductive hypothesis, there exist $z_1, \ldots, z_n \in S$ such that $J = (z_1, \ldots, z_n)R'$. Note there exists $e_i \in \mathbb{N}$ such that $z_i X_d^{e_i} \in I$ for $i = 1, \ldots, n$. Let $e = \max\{e_1, \ldots, e_n\}$. Then $z_i X_d^e \in I$ for $i = 1, \ldots, n$. Set $S_m = \{z \in [\![R']\!] \mid zX_d^m \in I\}$ and $J_m = (S_m)R'$ for $m = 0, \ldots, e-1$. (For $e = 0$, there are no S_m's nor J_m's to consider.) Similarly, for $m = 0, \ldots, e_1$, there exist $w_{m,1}, \ldots, w_{m,n_m} \in S_m$ such that $J_m = (w_{m,1}, \ldots, w_{m,n_m})R'$. Let

$$I' = (\{z_i X_d^e \mid i = 1, \ldots, n\} \cup \{w_{m,i} X_d^m \mid m = 0, \ldots, e-1; i = 1, \ldots, n_m\})\, R.$$

Then $I' \leqslant_m R$ is finitely generated by monomials from I and $I' \subseteq I$. Let $\underline{X}^{\underline{p}} = X_1^{p_1} \cdots X_d^{p_d} \in [\![I]\!]$. Assume $p_d \geqslant e$. Since $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \cdot X_d^{p_d} \in I$, $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \in S \subseteq J = (z_1, \ldots, z_n)R'$. So $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \in z_i R'$ for some $i \in \{1, \ldots, n\}$. Let $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} = z_i z$ for some $z \in R'$. Since $p_d \geqslant e$, $\underline{X}^{\underline{p}} = X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} X_d^{p_d} = z_i z X_d^e X_d^{p_d - e} = (z_i X_d^e)(z X_d^{p_d - e}) \in (z_i X_d^e)R \subseteq I'$. Assume $p_d < e$. Since $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \cdot X_d^{p_d} \in I$, $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \in S_{p_d} \subseteq J_{p_d} = (w_{p_d,1}, \ldots, w_{p_d,n_{p_d}})R'$. So $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \in w_{p_d,i} R'$ for some $j \in \{1, \ldots, n_{p_d}\}$. Let $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} = w_{p_d,j} w$ for some $w \in R'$. Since $p_d < e$, we have $w_{p_d,j} X_d^{p_d} \in I'$. Hence $\underline{X}^{\underline{p}} = X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} X_d^{p_d} = w_{p_d,i} w X_d^{p_d} = (w_{p_d,i} X_d^{p_d})(w) \in (w_{p_d,i} X_d^{p_d})R \subseteq I'$. Thus, in either case, $[\![I]\!] \subseteq I'$. Similarly, we have $I \subseteq I'$. □

Corollary 1.17. Let $S \subseteq [\![R]\!]$. If $I = (S)$, then there exist $s_1, \ldots, s_n \in S$ such that $I = (s_1, \ldots, s_n)$.

Proof. By Dickson's Lemma, there exists $i_1, \ldots, i_n \in I$ such that $(S) = I = \langle i_1, \ldots, i_n \rangle$. So $i_j = \sum_{k=1}^{n_j} s_{jk} r_{ik}$, where $r_{i1}, \ldots, r_{in_j} \in R$ for $j = 1, \ldots, n$. Hence $I = (\{s_{jk} \mid j = 1, \ldots, n, k = 1, \ldots, n_j\})$. □

Theorem 1.18. *(a) (ACC) Every ascending chain of monomials $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ must stablize, i.e., there $N \in \mathbb{N}$ such that $I_n = I_N$ for $n \geqslant N$.*

(b) Let $\Sigma \neq \emptyset$ be a set of monomial ideals in R. Then Σ contains at least one maximal element with respect to $\subseteq$. Moreover, for $I \in \Sigma$, there exists $J \in \Sigma$ maximal such that $I \subseteq J$.

Proof. (a) Let $I = \bigcup_{i=1}^{\infty} I_i$. Since the I_i's form a chain, $I \leqslant R$. Moreover, $I \leqslant_m R$ since $I = \sum_{i=1}^{\infty} I_i = \bigcup_{i=1}^{\infty} I_i = (\bigcup_{i=1}^{\infty} S_i)$, where S_i is a set of monomials generating I_i for each i. Then by Dickson's lemma, there exists $s_1, \ldots, s_n \in [\![I]\!]$ such that $I = (s_1, \ldots, s_n)$. Since $I = \bigcup_{i=1}^{\infty} I_i$, there exists p_j such that $s_j \in I_{p_j}$ for $j = 1, \ldots, n$. Let $p = \max(p_1, \ldots, p_n)$. Then $s_1, \ldots, s_n \in I_p \subseteq I$. So $I = (s_1, \ldots, s_n) \subseteq I_p \subseteq I_{p+1} \subseteq \cdots \subseteq I$, i.e., $I = I_p$. Thus, $I_p = I_{p+1} = I_{p+1} = \cdots$.

(b) Let $I \in \Sigma$. If I is maximal in Σ, then done. Assume there exists $I_1 \in \Sigma$ such that $I \subsetneq I_1$. If I_1 is maimal, then done. Otherwise, there exists $I_2 \in \Sigma$ such that $I \subsetneq I_1 \subsetneq I_2$. ACC implies process must terminate. So there exists $p \in \mathbb{N}$ such that $I \subseteq I_p$ with $I_p \in \Sigma$ maximal. Hence $\Sigma \neq \emptyset$. □

Definition 1.19. Let $z_1, \ldots, z_n \in [\![R]\!]$ and $I = (z_1, \ldots, z_n)$. The generating sequence $z_1, \ldots, z_n$ is *redundant* if there exists $i \in \{1, \ldots, n\}$ such that $I = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$. It is *irredundant* if it is not redundant, i.e., if for $i = 1, \ldots, n$, $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \subsetneq (z_1, \ldots, z_n)$.

Example 1.20. X^3, X^2Y, X^2Y^2, Y^5 is redundant since $(X^3, X^2Y, X^2Y^2, Y^5) = (X^3, X^2Y, Y^5)$.

Theorem 1.21. *$z_1, \ldots, z_m \in [\![R]\!]$ and $I = (z_1, \ldots, z_m)$. The followings are equivalent.*

(i) z_i is not a monomial multiple of z_j for $i, j = 1, \ldots, m$ with $i \neq j$.

(ii) For $i = 1, \ldots, m$, $z_i \notin (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$.

(iii) $z_1, \ldots, z_m$ is an irredundant monomial generating sequence for I.

Proof. (i)$\Longrightarrow$(ii) Assume (i). Suppose (ii) fails. Then there exists $i \in \{1, \ldots, m\}$ such that $z_i \in (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$. So $z_i \in z_j R$ for some $j \in \{1, \ldots, i-1, i+1, \ldots, m\}$ and then z_i is a monomial multiple z_j, a contradiction.

(ii)$\Longrightarrow$(iii) Note $z_i \in I \smallsetminus (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$. So $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m) \subsetneq I$ for $i = 1, \ldots, m$. Thus, it is irredundant.

(iii)$\Longrightarrow$(i) Suppose (i) fails. There exists $i, j \in \{1, \ldots, m\}$ with $i \neq j$ such that z_i is a monomial multiple of z_j. So $z_i \in (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$. Then $I \subseteq (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$. Also, $I \supseteq (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$. So $I = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$, a contradiction. □

Remark. Divisibility order on $[\![R]\!]$: $z, w \in [\![R]\!]$, $z \leqslant w$ if $z \mid w$, i.e., w is a monomial multiple of z. This is a partial order: reflexive, transitive and antisymmetric: if $z \mid w$ and $w \mid z$, then $w = z$.

$\mathbb{N}_0^d \to [\![R]\!]$ given by $\underline{n} \mapsto \underline{X}^{\underline{n}}$ is 1-1 and onto. $\underline{m} \succcurlyeq \underline{n}$ if and only if $\underline{X}^{\underline{n}} \mid \underline{X}^{\underline{m}}$, partial order for $\mathbb{N}_0^d$.

Remark (Criterion)**.** If $I = (f_1, \ldots, f_n) \leqslant_m R$, then $f_1, \ldots, f_n$ is irredundant if and only if $f_i \nmid f_j$ for $i, j = 1, \ldots, n$ with $i \neq j$.

Theorem 1.22. *Let $I \leqslant_m R$.*

(a) Every generating set $S \subseteq [\![I]\!]$ for I contains a finite irredundant monomial generating sequence.

(b) In particular, I have an irredundant monomial generating sequence.

(c) Irredundant monomial generating sequence for I is unique up to reordering.

Proof. (a) Assume without loss of generality, $S \neq \emptyset$. By Dickson's Lemma $I = (s_1, \dots, s_n)$ for some $s_1, \dots, s_n \in S$. If $s_1, \dots, s_n$ is irredundant, then done. If not, re-order $s_1, \dots, s_n$ to assume $I = (s_1, \dots, s_{n-1})$. If this is irredundant, then done, else, remove another generator. Process terminates in at most $n-1$ steps.

(b) By definition, I has a monomial generating set.

(c) Let $f_1, \dots, f_m$ and $g_1, \dots, g_n$ be two irredundant monoial generating sequences for I. Fix $i \in \{1, \dots, m\}$. Then $f_i \in (f_1, \dots, f_m) = I = (g_1, \dots, g_n)$. Since $f_i, g_1, \dots, g_n \in [\![R]\!]$, there exists $j \in \{1, \dots, n\}$ such that $f_i \in g_j R$. Similarly, there exists $k \in \{1, \dots, m\}$ such that $g_j \in f_k R$. So $f_k \mid g_j$ and $g_j \mid f_i$. Hence $f_k \mid f_i$. Since f's are irredundant, $k = i$. Then $f_i \mid g_j$ and $g_j \mid f_i$. So $f_i = g_j$. Define $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ by $i \mapsto j$ such that $f_i = g_j = g_{\sigma(i)}$. Suppose $g_j = f_i = g_l$, Since $g_1, \dots, g_n$ is irredundant, $l = j$ and then σ is well-defined. Similarly, since $f_1, \dots, f_m$ is irredundant, σ is 1-1. Reverse the process to get $\tau : \{1, \dots, n\} \to \{1, \dots, m\}$ is well-defined and 1-1. By PHP, σ and τ are bijections. □

Remark. Algorithms for finding an irredundant monomial generating sequence.

(a) Start with a finite monomial generating sequence $f_1, \dots, f_m$.

(b) If $f_i \nmid f_j$ for $i \neq j$, then done, else, $f_i \mid f_j$ for some $i \neq j$, let $I = (f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_m)$.

(c) Repeat with $f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_m$.

(d) Process terminates in at most m iterations.

Example 1.23. Since $(X^2Y, XY^3, X^2Y^2) = (X^2Y, XY^3)$, X^2Y, XY^3, X^2Y^2 is redundant.

Theorem 1.24. *Let $J = (S)$ with $\emptyset \neq S \subseteq [\![R]\!]$. Let $\Delta = \{\underline{n_z} \mid z = \underline{X}^{\underline{n_z}} \in S\} \subseteq \mathbb{N}_0^d$ and $\Delta' = \{$min. elts of Δ w.r.t $\succcurlyeq\}$.*

(a) $S' := \{\underline{X}^{\underline{n}} \mid \underline{n} \in \Delta'\}$ is an irredundant generating sequence for J.

(b) Thus, Δ' is finite.

Proof. Note Δ has minimal elements. by well-ordering axiom.

(a) Since $S' \subseteq S$, $(S') \subseteq (S) = J$. Let $z = \underline{X}^{\underline{n_z}} \in S$, then $\underline{n_z} \succcurlyeq \underline{n_w}$ for some $\underline{n_w} \in \Delta'$. So $\underline{X}^{\underline{n_z}} \in (\underline{X}^{\underline{n_w}}) \subseteq (S')$. Hence $S \subseteq (S')$ and so $(S) \subseteq (S')$. Thus, $(S') = (S) = J$. Then by previous theorem, $J = (z_1, \dots, z_m)$ for some irredundant generating sequence $z_1, \dots, z_m \in S'$. Let $z \in S' \subseteq J$, then $z \in z_j R$ for some $j \in \{1, \dots, m\}$. Since $z, z_j \in S'$ such that $z_j \mid z$, $z_j = z$. So $z \in \{z_1, \dots, z_m\}$. Thus, $\{z_1, \dots, z_m\} = S'$.

(b) Since S' is finite and Δ' is bijective with S', Δ' is finite. □

Chapter 2

Operations on Monomial Ideals

Let A be a nonzero commutative ring with identity and $R = A[X_1, \dots, X_d]$

2.1 Intersections

Theorem 2.1. *Let $I_1, \dots, I_n \leqslant_m R$. Then $I_1 \cap \cdots \cap I_n \leqslant_m R$ and is generated by $[\![I_1]\!] \cap \cdots \cap [\![I_n]\!]$. Also, $[\![I_1 \cap \cdots I_n]\!] = [\![I_1]\!] \cap \cdots \cap [\![I_n]\!]$.*

Proof. Let $J = (S)$, with $S := [\![I_1]\!] \cap \cdots \cap [\![I_n]\!] \subseteq I_1 \cap \cdots \cap I_n$. Then $J = (S) \subseteq I_1 \cap \cdots \cap I_n$. Let $f \in I_1 \cap \cdots \cap I_n$, with $f = \sum_{\underline{n} \in \mathbb{N}_0^d}^{\text{finite}} a_{\underline{n}} \underline{X}^{\underline{n}} \in I_j$ for $j = 1, \dots, n$. Since $I_j \leqslant_m R$, $\underline{X}^{\underline{n}} \in [\![I_j]\!]$ whenever $a_n \neq 0$ for $j = 1, \dots, n$. So $\underline{X}^{\underline{n}} \in [\![I_1]\!] \cap \cdots \cap [\![I_n]\!] = S$ whenever $a_n \neq 0$. Hence $f \in (S) = J$. Thus, $I_1 \cap \cdots \cap I_n = J = ([\![I_1]\!] \cap \cdots \cap [\![I_n]\!])$.

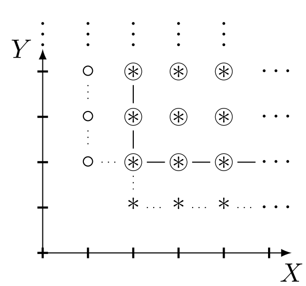
Note $[\![I_1 \cap \cdots I_n]\!] = (I_1 \cap \cdots \cap I_n) \cap [\![R]\!] = (I_1 \cap [\![R]\!]) \cap \cdots \cap (I_n \cap [\![R]\!]) = [\![I_1]\!] \cap \cdots \cap [\![I_n]\!]$. □

Remark. $\Gamma(I_1 \cap \cdots \cap I_n) = \Gamma(I_1) \cap \cdots \cap \Gamma(T_n)$.

Definition 2.2. Let $\underline{X}^{\underline{m}}, \underline{X}^{\underline{n}} \in [\![R]\!]$. Define $\mathrm{LCM}(\underline{X}^{\underline{m}}, \underline{X}^{\underline{n}}) = \underline{X}^{\underline{p}}$, where $p_i = \max(m_i, n_i)$ for $i = 1, \dots, d$.

Remark. If R is UFD, then always true for any polynomial.

Example 2.3. In $A[X, Y]$, to compute $(XY^2) \cap (XY^2)$, it suffices to compute $\Gamma((XY^2) \cap (X^2Y)) = \Gamma(\langle XY^2 \rangle) \cap \Gamma(\langle X^2Y \rangle)$ by previous remark.



Lemma 2.4. Let $f, g \in [\![R]\!]$. Then $(f) \cap (g) = (\mathrm{LCM}(f, g))$.

If R is UFD, then always true for any polynomial.

Proof. "$\supseteq$". Let $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}}$ for some $\underline{m}, \underline{n} \in \mathbb{N}_0^d$. Let $\underline{X}^{\underline{p}} = \mathrm{LCM}(f, g)$. Then $\underline{p} \succcurlyeq \underline{m}$ and $\underline{p} \succcurlyeq \underline{n}$. So $\underline{X}^{\underline{p}} \in (f)$ and $\underline{X}^{\underline{p}} \in (g)$.

"$\subseteq$". It suffices to show $[\![(f)]\!] \cap [\![(g)]\!] \subseteq (\mathrm{LCM}(f, g))$. Let $\underline{X}^{\underline{q}} \in [\![(f)]\!] \cap [\![(g)]\!]$. Then $\underline{X}^{\underline{q}} \in [\![(f)]\!], [\![(g)]\!]$. So $\underline{q} \succcurlyeq \underline{m}, \underline{n}$. Then $q_i \geqslant m_i, n_i$ for $i = 1, \ldots, d$. So $q_i \geqslant \max(m_i, n_i) = p_i$ for $i = 1, \ldots, d$, i.e., $\underline{q} \succcurlyeq \underline{p}$. Hence $\underline{X}^{\underline{q}} \in (\underline{X}^{\underline{p}}) = (\mathrm{LCM}(f, g))$. □

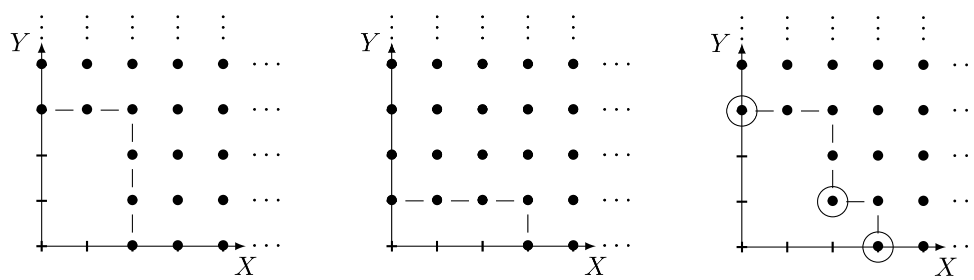
Theorem 2.5. *Let* $I = (f_1, \ldots, f_m)$ *and* $J = (g_1, \ldots, g_n)$ *with* $f_1, \ldots, f_m, g_1, \ldots, g_n \in [\![R]\!]$. *Then* $I \cap J = (\mathrm{LCM}(f_i, g_j) \mid i = 1, \ldots, m,\ j = 1, \ldots, n) := (K)$.

Proof. "$\subseteq$". Let $f \in [\![I]\!] \cap [\![J]\!]$. Then $f \in [\![I]\!], [\![J]\!]$. So $f \in (f_i), (g_j)$ for some $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Hence $f \in (f_i) \cap (g_j) = (\mathrm{LCM}(f_i, g_j)) \subseteq (K)$.

"$\supseteq$". Since $\mathrm{LCM}(f_i, g_j) \in (f_i) \cap (g_j) \subseteq I \cap J$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, we have $K \subseteq I \cap J$. So $(K) \subseteq I \cap J$. □

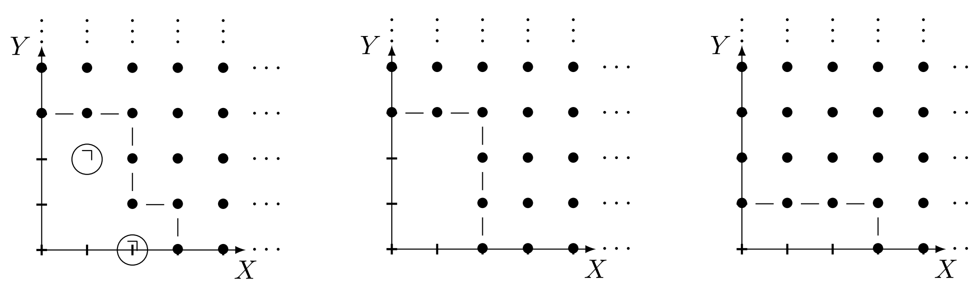
Example 2.6.

$$\begin{aligned} I := (X^2, Y^3) \cap (X^3, Y) &= \big(\mathrm{LCM}(X^2, X^3), \mathrm{LCM}(X^2, Y), \mathrm{LCM}(X^3, Y^3), \mathrm{LCM}(Y, Y^3)\big) \\ &= (X^3, X^2Y, X^3Y^3, Y^3) = (X^3, X^2Y, Y^5). \end{aligned}$$



$\Gamma(\langle X^2, Y^3 \rangle)$ $\Gamma(\langle X^3, Y \rangle)$ $\Gamma(I)$

One goal of this text is the following: given a monomial ideal I, to find simpler $I_1, \ldots, I_n \leqslant_m R$ such that $I = I_1 \cap \cdots \cap I_n$.



$\Gamma(I)$ $\Gamma(\langle X^2, Y^3 \rangle)$ $\Gamma(\langle X^3, Y \rangle)$

The two corners of the form ⌝ suggest the decomposition $I := (X^3, X^2Y, Y^5) = (X^3, Y^2) \cap (X^3, Y)$.

Lemma 2.7. $(f_1, \dots, f_m, gh) = (f_1, \dots, f_m, g) \cap (f_1, \dots, f_n, g)$, as long as g and h are "relative prime", i.e., $\mathrm{LCM}(f, g) = fg$.

2.2 Monomial ideals

Remark. Radical of a monomial ideal may not be a monomial ideal. Let $R = \frac{\mathbb{Z}}{8\mathbb{Z}}[x] \geqslant (x) =: I$. Since $2^3 = 0 \in I$, $2 \in \mathrm{rad}(I)$. So $\mathrm{rad}(I) = (2, x)$ is not a monomial ideal.

Definition 2.8. Define the *nilradical* of A by

$$\mathrm{Nil}(A) = \mathrm{rad}_A(0).$$

Definition 2.9. A ring A is *reduced* if $f^n = 0$ for some $n \in \mathbb{N}$ with $f \in A$, then $f = 0$.

Theorem 2.10. *A is reduced if and only if* $\mathrm{Nil}(A) = 0$.

Lemma 2.11. Let $f = a_0 + a_1x + \cdots a_mx^m \in A[x]$. Then $f^n = 0$ for some $n \in \mathbb{N}$ if and only if $a_i^k = 0$ for $i = 1, \dots, m$ and $k >> 0$. Or f is nilpotent if and only if coefficients of f are all nilpotent. Or $f \in \mathrm{Nil}(A[x])$ if and only if $a_1, \dots, a_m \in \mathrm{Nil}(A)$. Or $\mathrm{Nil}(A[x]) = \mathrm{Nil}(A)[x]$.

Proof. $\Longrightarrow$ Note $0 = f^n = (a_0 + a_1x + \cdots + a_mx^m)^n = a_0^n + \cdots + a_m^n x^{mn}$. So $a_m^n = 0$. Then $a_m \in \mathrm{Nil}(A) \subseteq \mathrm{Nil}(A[x]) \ni f$. Since $\mathrm{Nil}(A[x]) \leqslant A[x]$, $\mathrm{Nil}(A[x]) \ni f - a_mx^m = a_0 + \cdots + a_{m-1}x^{m-1}$. Induct on m to get $a_0, \dots, a_{m-1} \in \mathrm{Nil}(A)$. □

Example 2.12. If A is an integral domain, then A is reduced.

Definition 2.13. Let $I \leqslant_m R$. Then the *monomial radical* of I is

$$\text{m-rad}(I) = (\mathrm{rad}(I) \cap [\![R]\!])\,.$$

Remark. $\text{m-rad}(I) \leqslant_m R$ and $[\![\text{m-rad}(I)]\!] = \mathrm{rad}(I) \cap [\![R]\!]$.

Example 2.14. Let $R = A[X, Y]$. Then $\text{m-rad}\big((X^5, Y^7)\big) = (X, Y)$. "⊇". Done. "⊆". Since $1 \notin (X^5, Y^7)$, $1 \notin \mathrm{rad}\big((X^5, Y^7)\big)$. Note $\mathrm{rad}\big((X^5, Y^7)\big) \cap [\![R]\!] = \{X^aY^b \mid a \geqslant 1 \text{ or } b \geqslant 1\}$.

Example 2.15. $\text{m-rad}\big((X^2, XY)\big) = (X, XY) = (X)$.

Theorem 2.16. *Let $I \leqslant_m R$.*

(a) $\text{m-rad}(I) \subseteq \mathrm{rad}(I)$.

(b) $\text{m-rad}(I) = \mathrm{rad}(I)$ *if and only if* $\mathrm{rad}(I) \leqslant_m R$.

(c) If A is a field, then $\text{m-rad}(I) = \mathrm{rad}(I)$. *If A is reduced, then* $\text{m-rad}(I) = \mathrm{rad}(I)$.

Proof. (a) By definition of monomial radical.

(b) $\mathrm{rad}(I) = \text{m-rad}(I) \leqslant_m R$ if and only if $\mathrm{rad}(I) = (\mathrm{rad}(I) \cap [\![R]\!]) \leqslant_m R$.

(c) Assume A is reduced. Let $0 \neq f \in \operatorname{rad}(I)$. Let $w_1, \ldots, w_n \in [\![R]\!]$ be all the distinct monomials occurring in f. Then there exist $a_1, \ldots, a_n \in A \setminus \{0\}$ such that $f = \sum_{i=1}^n a_i w_i$. We use induction to show $w_1, \ldots, w_n \in \operatorname{rad}(I)$. Base case: $n = 1$ and $f = a_1 w_1 \in \operatorname{rad}(I)$. Then there exists $m \in \mathbb{N}$ such that $a_1^m w_1^m = f^m \in I$. If $a_1^m = 0$, then $a_1 = 0$ since A is reduced, contradicted by assumption $a_1 \neq 0$. So $a_1^m \neq 0$. Inductive step. Assume there exists $t \in \mathbb{N}$ such that $f^t \in I$. Reorder w_i's if necessary to assume $w_1 <_{\text{lex}} w_2 <_{\text{lex}} \cdots <_{\text{lex}} w_n$. Claim. the largest monomial occurring in f^t is w_n^t w.r.t. lex. Note f^t induces $(a_n w_n)^t = a_n^t w_n^t$. Since other monomials occuring in f^t have form $w_{i_1} \cdots w_{i_t}$, where $1 \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant n$ and $i_1 < n$, we have $w_{i_1} <_{\text{lex}} w_n$ and $w_{i_j} \leqslant_{\text{lex}} w_n$ for $j = 2, \ldots, t$. So $w_{i_1} \cdots w_{i_t} <_{\text{lex}} w_n^t$. Hence coefficients of w_n^t in f^t is a_n^t. Since A is reduced, $a_n^t \neq 0$. Also, since $f^t \in I \leqslant_m R$, $w_n^t \in I$. So $w_n \in \operatorname{rad}(I)$. Then $\sum_{i=1}^{n-1} a_i w_i = f_1 = f - a_n w_n \in \operatorname{rad}(I)$. By induction, $w_1, \ldots, w_{n-1} \in \operatorname{rad}(I)$. So $w_1, \ldots, w_n \in \operatorname{rad}(I)$. Thus, $\operatorname{rad}(I) \leqslant_m R$ by previous lemma. So $\text{m-rad}(I) = \operatorname{rad}(I)$ by (b). □

Theorem 2.17. *Let $I, J \leqslant_m R$.*

(a) $J \subseteq \text{m-rad}(J)$.

(b) $[\![\text{m-rad}(J)]\!] = [\![R]\!] \cap \operatorname{rad}(J)$.

(c) $I \subseteq J$ *implies* $\text{m-rad}(I) \subseteq \text{m-rad}(J)$.

(d) $\text{m-rad}(\text{m-rad}(J)) = \text{m-rad}(J)$.

(e) $\text{m-rad}(J) = R$ *if and only if* $J = R$.

(f) $\text{m-rad}(J) = 0$ *if and only if* $J = 0$.

(g) $\text{m-rad}(T^n) = \text{m-rad}(J)$.

Proof. (a) $J = (J \cap [\![R]\!]) \subseteq (\operatorname{rad}(J) \cap [\![R]\!]) = \text{m-rad}(J)$.

(b) "$\subseteq$". Since $[\![R]\!] \cap \operatorname{rad}(J) \subseteq ([\![R]\!] \cap \operatorname{rad}(J)) = \text{m-rad}(J)$ and $[\![R]\!] \cap \operatorname{rad}(J) \subseteq [\![R]\!]$, we have $[\![R]\!] \cap \operatorname{rad}(J) \subseteq [\![R]\!] \cap \text{m-rad}(J) = [\![m\text{-rad}(J)]\!]$. "$\supseteq$". $[\![m\text{-rad}(J)]\!] = [\![R]\!] \cap \text{m-rad}(J) \subseteq [\![R]\!] \cap \operatorname{rad}(J)$.

(c) By definition.

(d) "$\supseteq$". By (a), $\text{m-rad}(J) \subseteq \text{m-rad}(\text{m-rad}(J))$. "$\subseteq$". Let $f \in [\![\text{m-rad}(\text{m-rad}(J))]\!] = [\![R]\!] \cap \operatorname{rad}(\text{m-rad}(J))$. Then there exists $a \in \mathbb{N}$ such that $f^a \in \text{m-rad}(I) \cap [\![R]\!] = [\![\text{m-rad}(J)]\!] = [\![R]\!] \cap \operatorname{rad}(J)$. So there exists $b \in \mathbb{N}$ such that $f^{ab} = (f^a)^b \in J$. Thus, $f \in \operatorname{rad}(J) \cap [\![R]\!] = [\![\text{m-rad}(J)]\!]$. □

Theorem 2.18. *Let $I, J, I_1, \ldots, I_n \leqslant_m R$. Then*

(a) $\text{m-rad}(IJ) = \text{m-rad}(I \cap J) = \text{m-rad}(I) \cap \text{m-rad}(J)$.

(b) $\text{m-rad}(I_1 \cdots I_n) = \text{m-rad}(\cap_{i=1}^n I_i) = \cap_{i=1}^n \text{m-rad}(I_i)$.

(c) $\text{m-rad}(I + J) = \text{m-rad}(I) + \text{m-rad}(J)$.

(d) $\text{m-rad}(\sum_{i=1}^n I_i) = \sum_{i=1}^n \text{m-rad}(I_i)$.

Proof. (a) Since $\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J) = \operatorname{rad}(I) \cap \operatorname{rad}(J)$, $[\![R]\!] \cap \operatorname{rad}(IJ) = [\![R]\!] \cap \operatorname{rad}(I \cap J) = [\![R]\!] \cap \operatorname{rad}(I) \cap \operatorname{rad}(J) = ([\![R]\!] \cap \operatorname{rad}(I)) \cap ([\![R]\!] \cap \operatorname{rad}(J))$. So $[\![\text{m-rad}(IJ)]\!] = [\![\text{m-rad}(I \cap J)]\!] = [\![\text{m-rad}(I)]\!] \cap [\![\text{m-rad}(J)]\!]$.

(b) Induct on n.

(c) "$\supseteq$". Since $\mathrm{rad}(I+J)=\mathrm{rad}(\mathrm{rad}(I)+\mathrm{rad}(J))\supseteq \mathrm{rad}(I)+\mathrm{rad}(J)$, we have $[\![\text{m-rad}(I+J)]\!] = [\![R]\!]\cap \mathrm{rad}(I+J)\supseteq [\![R]\!]\cap(\mathrm{rad}(I)+\mathrm{rad}(J))$. So

$$\begin{aligned}[\![\text{m-rad}(I)+\text{m-rad}(J)]\!] &= [\![\text{m-rad}(I)]\!]\cup[\![\text{m-rad}(J)]\!] = ([\![R]\!]\cap\mathrm{rad}(I))\cup([\![R]\!]\cap\mathrm{rad}(J))\\ &= [\![R]\!]\cap(\mathrm{rad}(I)\cup\mathrm{rad}(J))\subseteq[\![R]\!]\cap(\mathrm{rad}(I)+\mathrm{rad}(J))\\ &\subseteq [\![\text{m-rad}(I+J)]\!].\end{aligned}$$

"$\subseteq$". Exercise.

(d) Induct on n. □

2.3 Generators of monomial ideals

Example 2.19. $\text{m-rad}(\langle X^3Y^2, XY^3, Y^5\rangle)=\langle XY, XY, Y\rangle=\langle Y\rangle$.

Definition 2.20. Let $f=\underline{X}^{\underline{n}}\in[\![R]\!]$. The *support* of f is

$$\mathrm{supp}(f)=\{i\in\mathbb{N}\mid n_i\geqslant 1\}=\{i\in\mathbb{N}\mid x_i\mid f\}.$$

The *reduction* of f is

$$\mathrm{red}(f)=\prod_{i\in\mathrm{supp}(f)}X_i=\prod_{X_i\mid f}X_i.$$

Example 2.21. $\mathrm{Supp}(X_1^5X_3^4)=\{1,3\}$ and $\mathrm{red}(X_1^5X_3^4)=X_1X_3$.

Lemma 2.22. Let $J\leqslant_m R$ and $f\in[\![R]\!]$.

(a) There exists $n\geqslant 1$ such that $\mathrm{red}(f)^n\in(f)$;

(b) If $f\in J$, then $\mathrm{red}(f)\in\text{m-rad}(J)$.

Proof. (a) Let $f=\underline{X}^{\underline{m}}$, then $n:=\max(m_1,\dots,m_d)\geqslant m_i$ for $i=1,\dots,d$. So $f\mid\mathrm{red}(f)^n$.

(b) Since $f\in J$, by (a), there exists $n\geqslant 1$ such that $\mathrm{red}(f)^n\in(f)\subseteq J$. So $\mathrm{red}(f)\in\text{m-rad}(J)$. □

Theorem 2.23. *Let* $S\subseteq[\![R]\!]$ *and* $J=\langle S\rangle$*, then* $\text{m-rad}(J)=\langle\mathrm{red}(s)\mid s\in S\rangle$.

Proof. "$\supseteq$". Let $s\in S$, then $s\in J$. By previous lemma, $\mathrm{red}(s)\in\text{m-rad}(J)$.

"$\subseteq$". Let $g\in[\![\text{m-rad}(J)]\!]=[\![R]\!]\cap\mathrm{rad}(J)$. Then $g^n\in J$ for some $n\in\mathbb{N}$. Since $g\in[\![R]\!]$, $g^n\in[\![J]\!]=[\![\langle S\rangle]\!]$. So there exists $s\in S$ such that $s\mid g^n$. Hence $\mathrm{red}(s)\mid\mathrm{red}(g^n)=\mathrm{red}(g)\mid g$. So $g\in\langle\mathrm{red}(s)\rangle\subseteq\langle\mathrm{red}(t\mid s\in S)\rangle$. Hence $[\![\text{m-rad}(J)]\!]\subseteq\langle\mathrm{red}(s)\mid s\in S\rangle$. □

Corollary 2.24. $\text{m-rad}(\langle X_{i_1}^{a_1},\dots,X_{i_t}^{a_t}\rangle)=\langle X_{i_1},\dots,X_{i_t}\rangle$.

2.4 Colon's of monomial ideals

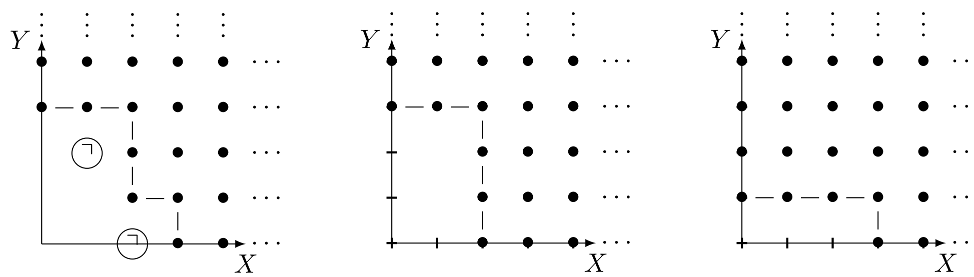
Theorem 2.25. *If $I, J \leqslant_m R$ are monomial ideals, then $(J : I) \leqslant_m R$.*

Proof. Special case: Let $I = zR$ with $z = \underline{X}^{\underline{m}} \in R$. Let $f = \sum_{\underline{n}\in\mathbb{N}^d}^{\text{finite}} a_{\underline{n}}\underline{X}^{\underline{n}} \in (J : z)$. Then $fz = \sum_{\underline{n}\in\mathbb{N}^d} a_{\underline{n}}\underline{X}^{\underline{n}+\underline{m}} \in J \leqslant_m R$. So $z\underline{X}^{\underline{n}} = \underline{X}^{\underline{n}+\underline{m}} \in J$ whenever $a_n \neq 0$, i.e., $\underline{X}^{\underline{n}} \in (J : zR)$ whenever $a_n \neq 0$. So $(J : z) \leqslant_m R$.

General case: Let $I = \langle s_1, \ldots, s_n\rangle = \sum_{i=1}^n \langle s_i\rangle$ for some $s_1, \ldots, s_n \in [\![I]\!]$. Then $(J : I) = (J : \sum_{i=1}^n \langle s_i\rangle) = \bigcap_{i=1}^n (J : \langle s_i\rangle) = \bigcap_{i=1}^n \langle J : s_i\rangle$. □

Remark. Let $f \in R = A[X, Y]$ and $\mathfrak{X} = \langle X, Y\rangle$. Then $f \in (I : \mathfrak{X})$ if and only if $f \in (I : X) \cap (I : Y)$ if and only if $Xf, Yf \in I$.

Example 2.26. Consider the ideal $I = (X^3, X^2Y, Y^3)$ with $R = A[X, Y]$ and $\mathfrak{X} = \langle X, Y\rangle$.



$\Gamma(I)$ $\Gamma(\langle X^2, Y^3\rangle)$ $\Gamma(\langle X^3, Y\rangle)$

The two corners of the form $\urcorner$ show us where to find elements of $(I : \mathfrak{X})$ not in I. It is not difficult to show that the monomials X^2 and XY^2 are precisely the monomials in $(I : \mathfrak{X}) \smallsetminus I$. Note that these "corners" corresponding to the "corners" in the ideals (X^2, Y^3) and (X^3, Y) in the decomposition $I = (X^2, Y^3) \cap (X^3, Y)$.

Remark (Notation)**.** Let $\underline{p}, \underline{q} \in \mathbb{N}^d$. Set $(\underline{p} - \underline{q})_i^+ = \begin{cases} p_i - q_i, & \text{if } p_i - q_i \geqslant 0 \\ 0, & \text{otherwise} \end{cases} = \max(p_i - q_i, 0)$ for $i = 1, \ldots, d$.

Example 2.27. $((1, 3) - (2, 1))^+ = (0, 2)$.

Theorem 2.28. $(\langle \underline{X}^{\underline{p}}\rangle : \langle \underline{X}^{\underline{q}}\rangle) = \langle \underline{X}^{(\underline{p}-\underline{q})^+}\rangle$.

Example 2.29. $(\langle X^3Y\rangle : \langle XY^2\rangle) = \langle X^2Y^0\rangle = \langle X^2\rangle$.

Theorem 2.30. *Let $I = \langle z_1, \ldots, z_n\rangle$ and $J = \langle w_1, \ldots, w_m\rangle$ with $z_i, w_j \in [\![R]\!]$. Then $(J : I) = \bigcap_{i=1}^n \left(\sum_{j=1}^m (\langle w_j\rangle : \langle z_i\rangle)\right)$.*

Proof. Note for $S \subseteq R$, $(I : \langle S\rangle) = (I : S)$. Case 1: $n = 1$. NTS $(J : I) = (\langle w_1, \ldots, w_m\rangle : z_1) = \sum_{j=1}^m (\langle w_j\rangle : z_1)$. "$\supseteq$". Let $f \in (\langle w_1\rangle : z_1)$. Then $fz_1 \in \langle w_1\rangle \subseteq J$. So $f \in (J : z_1)$. Then $(\langle w_1\rangle : z_1) \subseteq (J : z_1)$. Similarly, $(\langle w_1\rangle : z_j) \subseteq (J : z_1) \leqslant R$ for $j = 1, \ldots, m$. So $\sum_{j=1}^m (\langle w_j\rangle : z_1) \subseteq (J : z_1)$. "$\subseteq$". We have showed $(\langle w_j\rangle : z_1) \leqslant_m R$ for $j = 1, \ldots, m$. So

$\sum_{j=1}^{m}(\langle w_j\rangle : z_1) \leqslant_m R$. Let $f \in [\![R]\!]$ such that $fz_1 \in J = \langle w_1, \ldots, w_m\rangle$. So $fz_1 \in \langle w_j\rangle$ for some $j \in \{1, \ldots, m\}$. Then $f \in (\langle w_j\rangle : z_1)$. So $f \in \sum_{j=1}^{m}(\langle w_j\rangle : z_1)$. Since $I, J \leqslant_m R$, $(J : I) \leqslant_m R$. Hence $(J : I) \subseteq \sum_{j=1}^{m}(\langle w_j\rangle : z_1)$.

General case:

$$(J : I) = (\langle w_1, \ldots, w_m\rangle : \langle z_1, \ldots, z_n\rangle) = \left(\langle w_1, \ldots, w_m\rangle : \sum_{i=1}^{n}\langle z_i\rangle\right)$$
$$= \bigcap_{i=1}^{n}(\langle w_1, \ldots, w_m\rangle : z_i) = \bigcap_{i=1}^{n}\left(\sum_{j=1}^{m}(\langle w_j\rangle : z_i)\right). \qquad \square$$

Example 2.31. $(\langle X^3, Y^4\rangle : \langle X^2Y, XY^2\rangle) = \big[(\langle X^3\rangle : \langle X^2Y\rangle) + (\langle Y^4\rangle : \langle X^2Y\rangle)\big] \bigcap \big[(\langle X^3\rangle : \langle XY^2\rangle) + (\langle Y^4\rangle : \langle XY^2\rangle)\big] = \big[\langle X\rangle + \langle Y^3\rangle\big] \cap \big[\langle X^2\rangle + \langle Y^2\rangle\big] = \langle X, Y^3\rangle \cap \langle X^2, Y^2\rangle = \langle X^2, XY^2, X^2Y^3, Y^3\rangle = \langle X^2, XY^2, Y^3\rangle$.

Theorem 2.32. *Let $I = \langle g_1, \ldots, g_t\rangle \leqslant_m R$. Let $h \in [\![R]\!]$, then $(I : h) = \langle \frac{g_1}{\gcd(g_1,h)}, \ldots, \frac{g_t}{\gcd(g_t,h)}\rangle$.*

2.5 Bracket powers of monomial ideals

We know the power I^n of an ideal I is not generated by the n^{th} powers of the generators of I. This section investigates the ideal that is generated by powers of the generator of I.

Definition 2.33. Let $I \leqslant_m R$. The k^{th} *bracket power* of I is the ideal $I^{[k]} = (\{f^k \mid f \in [\![I]\!]\})$ for $k \in \mathbb{N}$.

Remark. By definition, $J^{[k]} \leqslant_m R$ for $k \in \mathbb{N}$.

Lemma 2.34. Let $S \subseteq [\![R]\!]$ and $k \in \mathbb{N}$. Set $I = (S)$ and $J = (\{f^k \mid f \in S\})$. If $g \in [\![R]\!]$, then $g \in I$ if and only if $g^k \in J$.

Proof. $\Longrightarrow$ Assume $g \in I$. Since $I \leqslant_m R$, by Dickson's lemma, there exists a finite $S' \subseteq S$ such that $I = (S')$. So $g \in (f)$ for some $f \in S' \subseteq S$. Hence $g^k \in (f^k) \subseteq J$.

$\Longleftarrow$ Assume $g^k \in J$. Since $J \leqslant_m R$, there exists a finite $S'_k \subseteq S_k := \{f^k \mid f \in S\}$ such that $J = (S'_k)R$ by Dickson's lemma. So $g^k \in (f^k)$ for some $f^k \in S'_k$ with $f \in S$. Write $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}}$ with $m, n \in \mathbb{N}_0^d$. Then $k\underline{m} \succcurlyeq k\underline{n}$, i.e., $\underline{m} \succcurlyeq \underline{n}$. So $g = \underline{X}^{\underline{n}} \in (\underline{X}^{\underline{m}}) = (f) \subseteq (S) = I$. $\square$

Proposition 2.35. Let $I \leqslant_m R$.

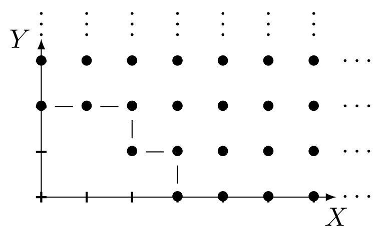
(a) If $S \subseteq [\![R]\!]$ and $I = (S)$, then $I^{[k]} = (\{f^k \mid f \in S\})$.

(b) If $I = (f_1, \ldots, f_n) \leqslant_m R$, then $I^{[k]} = (f_1^k, \ldots, f_n^k)$.

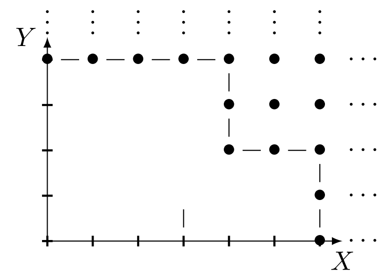
Proof. (a) By definition of $I^{[k]}$, $I^{[k]} \supseteq (\{f^k \mid f \in S\}) =: J$. Let $g \in [\![R]\!] \cap I = [\![I]\!]$. Then by previous lemma, $g^k \in J = (\{f^k \mid f \in S\})$. Note g^k is an arbitrary genertor of $I^{[k]}$. So $I^{[k]} \subseteq (\{f^k \mid f \in S\})$.

(b) This is the special case of (a). $\square$

Example 2.36. Let $I := (X^3, X^2Y, Y^2) \leqslant A[X, Y]$. Then $I^{[2]} = (X^6, X^4Y^2, Y^4)$.



$\Gamma(\langle X^3, X^2Y, Y^2\rangle)$



$\Gamma(\langle X^6, X^4Y^2, Y^4\rangle)$

Lemma 2.37. Let $I \leqslant_m R$, $g \in [\![R]\!]$ and $k \in \mathbb{N}$. Then $g \in I$ if and only if $g^k \in I^{[k]}$.

Proof. It follows from previous lemma and proposition. □

Proposition 2.38. Let $I \leqslant_m R$. Let $f_1, \ldots, f_n \in [\![I]\!]$ be an irredundant generating sequence for I and $k \in \mathbb{N}$. Then $I^{[k]}$ is irredundantly generated by $f_1^k, \ldots, f_n^k$.

Proof. By previous proposition, $f_1^k, \ldots, f_n^k \in [\![I]\!]$ is a generating sequence for $J^{[k]}$. Suppose $f_1^k, \ldots, f_n^k$ is redundant. Then there exsits $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $f_i^k \in (f_j^k) = (f_j)^{[k]}$. So by previous lemma, $f_i \in (f_j)$, a contradiction. □

Lemma 2.39. Let $I, J \leqslant_m R$ and $k \in \mathbb{N}$.

(a) $I \subseteq J$ if and only if $I^{[k]} \subseteq J^{[k]}$.

(b) $I = J$ if and only if $I^{[k]} = J^{[k]}$.

Proof. (a) $\Longrightarrow$ By definition of $I^{[k]}$ and $J^{[k]}$.

$\Longleftarrow$ Assume $I^{[k]} \subseteq J^{[k]}$. Let $g \in [\![I]\!]$. Then $g^k \in I^{[k]} \subseteq J^{[k]}$. So $g \in J$ by previous lemma.

(b) It follows from (a). □

Since ther intersection of monomial ideals is a monomial ideal, $(\bigcap_{i=1}^n J_i)^{[k]}$ is defined.

Proposition 2.40. Let $J_1, \ldots, J_n \leqslant_m R$ and $k \in \mathbb{N}$. Then $(\bigcap_{i=1}^n J_i)^{[k]} = \bigcap_{i=1}^n J_i^{[k]}$.

Proof. By induction. □

Chapter 3

M-Irreducible Ideals and Decompositions

Let A be a nonzero commutative ring with identity and $R = A[X_1, \ldots, X_d]$.

3.1 M-irreducible monomial ideals

Definition 3.1. $I \leqslant_m R$ is *m-reducible* if there exist $J, K \leqslant_m R$ such that $I = J \cap K$ and $J \neq I \neq K$.

$I \lneq_m R$ is *m-irreducible* if it is not m-reducible.

Theorem 3.2. *If A is a field and $I \leqslant_m R$ is m-irreducible, then I is irreducible.*

Theorem 3.3. *$I \leqslant_m R$ is m-irreducible if and only if $I \neq R$ and for any $J, K \leqslant_m R$, if $I = J \cap K$, then $I = J$ or $I = K$.*

Example 3.4. (X^3, X^2Y^2, Y^4) is m-reducible since $(X^3, Y^2) \cap (X^2, Y^4) = (X^3, X^2Y^2, Y^4)$, and $Y^2 \in (X^3, Y^2) \smallsetminus \big((X^3, X^2Y^2, Y^4) \cup (X^2, Y^4)\big)$, $X^2 \in (X^2, Y^4) \smallsetminus \big((X^3, X^2Y^2, Y^4) \cup (X^3, Y^2)\big)$.

Theorem 3.5. *$0 \neq I \leqslant_m R$ is m-irreducible if and only if it is generated by "pure powers", i.e., if and only if $I = \langle X_{i_1}^{a_1}, \ldots, X_{i_t}^{a_t} \rangle$ for some $t \geqslant 1$ and $a_i \geqslant 1$ for $i = 1, \ldots, t$.*

Proof. "$\Leftarrow$". Reorder $X_{i_1}, \ldots, X_{i_t}$ if necessary to assume $I = \langle X_1^{a_1}, \ldots, X_t^{a_t} \rangle \subseteq \langle X_1, \ldots, X_d \rangle \subsetneq R$. Suppose there exist $J, K \leqslant_m R$ such that $I = J \cap K$ and $J \neq I \neq K$. Then $I \subsetneq J, K$. So $[\![I]\!] \subsetneq [\![J, [\![K]\!]$. Let $f = \underline{X}^{\underline{m}} \in [\![J]\!] \smallsetminus [\![I]\!]$ and $g = \underline{X}^{\underline{n}} \in [\![K]\!] \smallsetminus [\![I]\!]$. Let $\underline{X}^{\underline{p}} := \mathrm{LCM}(f, g)$. Since $f \notin [\![I]\!]$, $X_i^{a_i} \nmid \underline{X}^{\underline{m}}$, i.e., $a_i > m_i$ for $i = 1, \ldots, t$. Similarly, $a_i > n_i$ for $i = 1, \ldots, t$. So $a_i > \max(m_i, n_i) = p_i$, i.e., $X_i^{a_i} \nmid \underline{X}^{\underline{p}}$ for $i = 1, \ldots, t$. Hence $\underline{X}^{\underline{p}} \notin \langle X_1^{a_1}, \ldots, X_t^{a_t} \rangle = I$. Since $f \in [\![J]\!]$ and $g \in [\![K]\!]$ and $J, K \leqslant_m R$, $(\mathrm{LCM}(f, g))R = (f) \cap (g) \subseteq ([\![J]\!]) \cap ([\![K]\!]) = J \cap K$. So $\underline{X}^{\underline{p}} = \mathrm{LCM}(f, g) \in J \cap K = I$, a contradiction.

$\Longrightarrow$ Take an irredundant monomial generating sequence $f_1, \ldots, f_k$ for I, where at least one of the f_i's is not a pure power. Reorder the f_i's to assume f_k is not a pure power. Then there exists $j \in \{1, \ldots, n\}$ such that $f_k = X_j^{c_j} g$, where $c_j \geqslant 1, X_j \nmid g$ and $g \neq 1$. Reorder the variables if necessary to assume $j = 1$. Then $f_k = X_1^{c_1} g$, where $c_1 \geqslant 1, X_1 \nmid g$ and $g \neq 1$. Set $J = \langle f_1, \dots, f_{k-1}, X_1^{c_1} \rangle$, $K = \langle f_1, \dots, f_{k-1}, g\rangle$. Then

$$\begin{aligned} J \cap K &= \langle \mathrm{LCM}(f_1, f_1), \mathrm{LCM}(f_2, f_2), \dots, \mathrm{LCM}(f_{k-1}, f_{k-1}), \mathrm{LCM}(X_1^{c_1}, g), \cdots \rangle \\ &= \langle f_1, f_2, \dots, f_{k-1}, X_1^{c_1} g, \cdots \rangle = \langle f_1, f_2, \dots, f_{k-1}, X_1^{c_1} g \rangle = \langle f_1, f_2, \dots, f_{k-1}, f_k \rangle = I. \end{aligned}$$

Suppose $X_1^{c_1} \in I$. Then there exists $i \in \{1, \dots, k\}$ such that $f_i \mid X_1^{c_1} \mid f_k$, i.e., $f_i \mid f_k$. Since $f_1, \dots, f_n$ is an irredundant monomial generating sequence, $f_k = f_i \mid X_1^{c_1}$, a contradiction. So $X_1^{c_1} \notin I$. Hence $I \subsetneq K$. Similarly, $I \subsetneq K$. □

Example 3.6. (X^3, X^2Y^2, Y^4) is not m-irreducible. (X^2, Y^4) and (X^3, Y^2) are both m-irreducible.

Theorem 3.7. *If $I, J_1, \dots, J_n \leqslant_m R$ such that I is m-irreducible and $I \supseteq \bigcap_{i=1}^n J_i$, then $I \supseteq J_i$ for some $i \in \{1, \dots, n\}$.*

Proof. If $I = 0$, then $I = 0 \supseteq \bigcap_{i=1}^n J_i$, so $\bigcap_{i=1}^n J_i = 0$. Then $J_1 \cdots J_n \subseteq \bigcap_{i=1}^n J_i = 0$. So there exists $i \in \{1, \dots, n\}$ such that $J_i = 0 = I$. Assume $I \neq 0$. Assume $n \geqslant 2$. Induct on n. Let's show $n = 2$. Let $\langle X_{i_1}^{a_1}, \dots, a_{i_t}^{a_t} \rangle = I \supseteq J_1 \cap J_2$. Suppose $I \not\supseteq J_1, J_2$. Then $[\![I]\!] \not\supseteq [\![J_1]\!], [\![J_2]\!]$. Let $f_1 = \underline{X}^{\underline{m}} \in [\![J_1]\!] \smallsetminus [\![I]\!]$ and $f_2 = \underline{X}^{\underline{n}} \in [\![J_2]\!] \smallsetminus [\![I]\!]$. Let $\underline{X}^{\underline{p}} := \mathrm{LCM}(f_1, f_2) = (f_1) \cap (f_2) \subseteq J_1 \cap J_2 \subseteq \langle X_{i_1}^{a_1}, \dots, X_{i_t}^{a_t} \rangle$. Then $X_{i_j}^{a_j} \mid \underline{X}^{\underline{p}}$ for some $j \in \{1, \dots, t\}$. So $a_j \leqslant p_{i_j} = \max(m_{i_j}, n_{i_j}) \leqslant m_{i_j}, n_{i_j}$, i.e., $X_{i_j}^{a_j} \mid \underline{X}^{\underline{m}} = f_1$ or $X_{i_j}^{a_j} \mid \underline{X}^{\underline{n}} = f_2$ for $j = 1, \dots, t$. Hence $f_1 \in \langle X_{i_1}^{a_1}, \dots, X_{i_t}^{a_t} \rangle = I$ or $f_2 \in \langle X_{i_1}^{a_1}, \dots, X_{i_t}^{a_t} \rangle = I$. Thus, $f_1 \in [\![I]\!]$ or $f_2 \in [\![I]\!]$, a contradiction. □

3.2 M-irreducible decomposition

Definition 3.8. An *m-irreducible decomposition* of $I \lneqq_m R$ is an expression $I = \bigcap_{i=1}^n J_i$ with $n \geqslant 1$ such that $J_1, \dots, J_n \leqslant_m R$ are m-irreducible.

Remark. 0 intersection of ideals is R.

Example 3.9. $(X^2, XY, Y^3) = (X, Y^3) \cap (X^2, Y)$ is an m-irreducble decomposition.

Theorem 3.10. *$I \lneqq_m R$ has an m-irreducible decomposition.*

Proof. Suppose not. Let $\Sigma = \{J \lneqq_m R \mid J$ doesn't has an m-irreducble decomposition$\}$. Then $\Sigma \neq \emptyset$. By previous theorem, Σ has a maximal element I_1, not m-irreducible. So there exists $J, K \leqslant_m R$ such that $I_1 = J \cap K$ and $I_1 \subsetneq J, K$. Since I_1 is maximal in Σ, $J, K \notin \Sigma$. Since $J, K \lneqq_m R$, they have an m-irreducible decomposition. So $I_1 = J \cap K$ has an m-irreducble decomposition, a contradiction. □

Definition 3.11. An m-irreducible decomposition $I = \bigcap_{i=1}^n J_i$ is *redundant* if $I = \bigcap_{i \neq k}^n J_i$ for some $k \in \{1, \dots, n\}$.

Theorem 3.12. *Given an m-irreducible decomposition $I = \bigcap_{i=1}^n J_i$. The followings are equivalent.*

(i) The decomposition is redundant.

(ii) There exists $i, j \in \{1, \dots, n\}$ with $i \neq j$ such that $J_i \subseteq J_j$.

Proof. "(ii)⇒(i)". If $J_i \subseteq J_j$, then $I = \bigcap_{k=1}^n J_k = \bigcap_{k\neq j}^n J_k$.

"(i)⇒(ii)". If the decomposition is redundant, then there exists $j \in \{1,\dots,n\}$ such that $J_j \supseteq \bigcap_{i=1}^n J_i = I = \bigcap_{k\neq j}^n J_k$. By previous theorem, there exists $k \in \{1,\dots,n\}$ with $k \neq j$ such that $J_j \supseteq J_k$. □

Theorem 3.13. *$I \lneq_m R$ has an irredundant m-irreducble decomposition.*

Proof. Let $I = \bigcap_{i=1}^n J_i$ with $n \geqslant 1$ be an m-irreducible decomposition. If it is irredundant, stop. Else, there exists $j \in \{1,\dots,n\}$ such that $I = \bigcap_{i\neq j}^n J_i$. Repeat with the new decomposition. Process terminate in at most $n-1$ steps. □

Theorem 3.14 (Irredundant m-irreducible decompositions are unique up to reodering)**.** *Let $I \leqslant_m R$ with two irredundant m-irreducible decompositions $\bigcap_{i=1}^s I_i = \bigcap_{j=1}^t I_j$. Then $s = t$ and there exists $\sigma \in S_t$ such that $I_i = J_{\sigma(i)}$ for $i = 1,\dots,t$.*

Proof. Let $i \in \{1,\dots,s\}$ be given. Then $\bigcap_{i=1}^s I_i = \bigcap_{j=1}^t I_j \subseteq I_i$. So there exists $j \in \{1,\dots,t\}$ such that $J_j \subseteq I_i$. Similarly, there exists $k \in \{1,\dots,s\}$ such that $I_k \subseteq J_j \subseteq I_i$. Since $\bigcap_{l=1}^s I_l$ is irredundant, $k = i$ and then $J_j = I_i$. Suppose there exists $m \in \{1,\dots,t\}$ such that $J_m = I_i = J_j$. Since $\bigcap_{l=1}^t J_l$ is irredundant, $j = m$. So there exists a unique $j \in \{1,\dots,t\}$ such that $J_j = I_i$. Define $\sigma : [s] \to [t]$ by $i \mapsto$ unique j such that $J_j = I_i = J_{\sigma(i)}$. By symmetry, there exists $\tau : [t] \to [s]$ given by $j \mapsto$ unique k such that $I_k = J_j = I_{\tau(j)}$. Check $\sigma \circ \tau = \mathrm{id}$ and $\tau \circ \sigma = \mathrm{id}$. □

Remark. The "splitting generators" algorithm can be established using the previous proof. Assume I has a m-reducible monomial generators $f_1 = X_1^{e_1} g$, $e_1 \geqslant 1$ and $X_1 \nmid g$. Then decompose I as $I = \langle f_1,\dots,f_n\rangle = \langle X_1^{e_1}, f_2,\dots,f_n\rangle \cap \langle g, f_2,\dots,f_n\rangle$.

Example 3.15.

$$\begin{aligned}\langle X^3YZ, XY^4Z\rangle &= \langle X^3, XY^4Z\rangle \cap \langle Y, XY^4Z\rangle \cap \langle Z, XY^4Z\rangle \\ &= \langle X^3,X\rangle \cap \langle X^3,Y^4\rangle \cap \langle X^3,Z\rangle \cap \langle Y,X\rangle \cap \langle Y\rangle \cap \langle Y,Z\rangle \cap \langle Z,X\rangle \cap \langle Z,Y^4\rangle \cap \langle Z\rangle \\ &= \langle X\rangle \cap \langle Y\rangle \cap \langle Z\rangle \cap \langle X^3,Y^4\rangle.\end{aligned}$$

Chapter 4

Connections with Combinatorics

Let A be a nonzero commutative ring with identity and $R = A[X_1, \ldots, X_n]$.

4.1 Square free monomial ideals

Definition 4.1. A monomial $\underline{X}^{\underline{n}}$ is *square-free* if $n_i \leqslant 1$, i.e., $X_i^2 \nmid \underline{X}^{\underline{n}}$ for $i = 1, \ldots, d$, i.e., $\underline{X}^{\underline{n}} = \text{red}(\underline{X}^{\underline{n}})$.

$I \leqslant_m R$ is *square-free* if it is generated by square free monomials.

Example 4.2. $\langle X^3YZ, XY^4Z\rangle$ is not square-free.

Theorem 4.3. *$I \leqslant R$ is square-free if and only if irredundant monomial generating sequence is square-free.*

Example 4.4. 0 is square-free since $0 = \langle \emptyset \rangle$ and $\emptyset \subseteq \{\text{square-free monomials}\}$.

Theorem 4.5. *$J \leqslant_m R$ is square-free if and only if $J = \text{m-rad}(J)$ if and only if $J = \text{m-rad}(I)$ for some $I \leqslant_m R$. In particular, if $I \leqslant_m R$, then $\text{m-rad}(I)$ is square-free.*

Proof. Assume first that J is square-free. Then J has a square-free monomial generating sequence. Let $f_1, \ldots, f_n \in [\![J]\!]$ be a irredundant generating sequence for J. Then by previous theorem, the square-free monomial generating sequence contains the f_i's. So $f_1, \ldots, f_n$ is square-free. Hence $\text{m-rad}(J) = \langle \text{red}(f_1), \ldots, \text{red}(f_n)\rangle = \langle f_1, \ldots, f_n\rangle = J$.

Assume $J = \text{rad}(I)$ for some $I \leqslant_m R$. Let $g_1, \ldots, g_m \in [\![I]\!]$ be a generating sequence for I. Then $J = \text{m-rad}(I) = \langle \text{red}(g_1), \ldots, \text{red}(g_n)\rangle$. So J is square-free. □

Remark. Assume A is a field. We know then $\text{m-rad}(I) = \text{rad}(I)$. So J is square-free if and only if $J = \text{rad}(J)$ if and only if $J = \text{rad}(I)$ for some $I \leqslant_m R$.

Theorem 4.6. *Let $I \leqslant_m R$. Then I is square-free and m-irreducible if and only if $I = \langle X_{i_1}, \ldots, X_{i_t}\rangle$ for some $t \geqslant 1$ and $i_1, \ldots, i_t \in \{1, \ldots, d\}$.*

Proof. $\Longrightarrow$ Assume I is square-free and m-irreducible. Since I is m-irreducible, there exists an irredundant monomial generating sequence $X_{i_1}^{a_1}, \ldots, X_{i_t}^{a_t}$ with $a_1, \ldots, a_t \geqslant 1$. Since I is square-free, by previous theorem, $I = \text{m-rad}(I) = \langle X_{i_1}, \ldots, X_{i_t}\rangle$.

$\Longleftarrow$ It is similar. □

Theorem 4.7. *Let $J = \bigcap_{i=1}^{n} J_i$ be an m-irreducible decomposition.*

(a) If $J_1, \ldots, J_n$ are square-free, so is J.

(b) If J is square-free and the intersection is irredundant, then $J_1, \ldots, J_n$ are square-free.

Proof. (a) Since $J_1, \ldots, J_n$ are square-free, $J_i = \text{m-rad}(J_i)$ for $i = 1, \ldots, n$. So $\text{m-rad}(J) = \text{m-rad}\left(\bigcap_{i=1}^{n} J_i\right) = \bigcap_{i=1}^{n} \text{m-rad}(J_i) = \bigcap_{i=1}^{n} J_i = J$.

(b) Assume J is square-free. Let $k \in \{1, \ldots, n\}$. Then $J_k \supseteq \bigcap_{i=1}^{n} J_i = J = \text{m-rad}(J) = \bigcap_{i=1}^{n} \text{m-rad}(J_i)$. So $J_k \supseteq \text{m-rad}(J_i) \supseteq J_i$ for some $i \in \{1, \ldots, n\}$. Since the decomposition is irredundant, $k = i$. So $J_k = \text{m-rad}(J_i) = J_i$. Then $J_k = \text{m-rad}(J_i) = \text{m-rad}(J_k)$. □

Example 4.8. $\langle X^3YZ, XY^4Z\rangle = \langle X\rangle \cap \langle Y\rangle \cap \langle Z\rangle \cap \langle X^3, Y^4\rangle$, is not square free.

Example 4.9. $\langle XYZ, YZW\rangle = \langle Y\rangle \cap \langle Z\rangle \cap \langle X, W\rangle$ is square-free.

Definition 4.10. Let $V = \{v_1, \ldots, v_d\}$ and $V' \subseteq V$. Define

$$P_{V'} = \langle X_i \mid v_i \in V'\rangle.$$

Example 4.11. $P_{v_1,v_3} = \langle X_1, X_3\rangle$ and $P_\emptyset = \langle\emptyset\rangle = 0$.

Theorem 4.12. *$I \lneq_m R$ is square-free if and only if there are $V_1, \ldots, V_n \subseteq V$ such that $J = \bigcap_{i=1}^{n} P_{V_i}$.*

Proof. By previous theorems. □

4.2 Polarization

Definition 4.13. Let $\underline{X}^{\underline{a}} \in [\![R]\!]$. Define the *polarization* of M to be the square-free monomial

$$\mathcal{PO}(\underline{X}^{\underline{a}}) = X_{1,1} \cdots X_{1,a_1} X_{2,1} \cdots X_{2,a_2} \cdots X_{d,1} \cdots X_{d,a_d}$$

in the polynomial ring $S = A[X_{i,j} \mid 1 \leqslant i \leqslant d, 1 \leqslant j \leqslant a_i]$.

Let $I = (\underline{X}^{\underline{a}_1}, \ldots, \underline{X}^{\underline{a}_n}) \leqslant_m R$. Define the *polarization* of I by

$$\mathcal{PO}(I) = (\mathcal{PO}(\underline{X}^{\underline{a}_1}), \ldots, \mathcal{PO}(\underline{X}^{\underline{a}_n})).$$

Example 4.14. Let $(X_1^2, X_1X_2, X_2^3) \subseteq A[X_1, X_2]$. Then $\mathcal{PO} = (X_{1,1}X_{1,2}, X_{1,1}X_{2,1}, X_{2,1}X_{2,2}X_{23})$ in $A[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}, X_{2,3}]$.

Remark. By identifying each X_i with $X_{i,1}$, one can consider S as a polynomial extension of R.

Let A be a field.

Proposition 4.15. Let $I, J \leqslant_m R$.

(a) $\mathcal{PO}(I + J) = \mathcal{PO}(I) + \mathcal{PO}(J)$.

(b) Let $f, g \in [\![R]\!]$. Then $f \mid g$ if and only if $\mathcal{PO}(f) \mid \mathcal{PO}(g)$.

(c) $\mathcal{PO}(I \cap J) = \mathcal{PO}(I) \cap \mathcal{PO}(J)$.

(d) If $\mathfrak{p}$ is a (minimal) prime containing I, then $\mathcal{PO}(\mathfrak{p})$ is a (minimal) prime containing $\mathcal{PO}(I)$.

(e) $\operatorname{ht}(I) = \operatorname{ht}(\mathcal{PO}(I))$ in the corresponding ring, respectively.

Proposition 4.16 (Froberg)**.** Let $\underline{X}^{\underline{a}_1}, \dots, \underline{X}^{\underline{a}_n} \in [\![R]\!]$. Let $m_j = \max_{1 \leqslant i \leqslant n}\{a_{i,j}\}$ for $j = 1, \dots, d$. Let $N_1 = \mathcal{PO}(\underline{X}^{\underline{a}_1})$, $\cdots$, $N_n = \mathcal{PO}(\underline{X}^{\underline{a}_n})$ in $S := A[X_{i,k} \mid 1 \leqslant i \leqslant d, 1 \leqslant k \leqslant m_i]$ such that X_{i,m_i} appears in at least one of the monomials $N_1, \dots, N_n$ for $i = 1, \dots, d$. Then the sequence of elements $X_{i,1} - X_{i,k}, 1 \leqslant i \leqslant d, 2 \leqslant k \leqslant m_i$ forms a regular sequence in $R' := \frac{S}{(N_1, \dots, N_n)}$. Let $I = (\{X_{i,1} - X_{i,k} \mid 1 \leqslant i \leqslant d, 2 \leqslant k \leqslant m_i\}) \leqslant R'$. Then

$$\frac{k[X_1, \dots, X_d]}{(\underline{X}^{\underline{a}_1}, \dots, \underline{X}^{\underline{a}_n})} = \frac{R}{(\underline{X}^{\underline{a}_1}, \dots, \underline{X}^{\underline{a}_n})} = \frac{R'}{I} = \frac{A[X_{i,k} \mid 1 \leqslant i \leqslant d, 1 \leqslant k \leqslant m_i]}{(N_1, \dots, N_n) + (\{X_{i,1} - X_{i,k} \mid 1 \leqslant i \leqslant d, 2 \leqslant k \leqslant m_i\})}.$$

Moreover, R is Cohen-Macaulay (Gorenstein) if and only if R' is.

Example 4.17. Let $H = (V_H, E_H)$ be a suspension of G with $V_H = \{V_G\} \cup \{w_1, \dots, w_d\}$. Then $K(\Sigma G) := \frac{k[X_1, \dots, X_d, Y_1, \dots, X_d]}{I(H)} = \frac{k[X_1, \dots, X_d, Y_1, \dots, Y_d]}{(I(G) + \langle X_1Y_1, \dots, X_dY_d\rangle)}$. So $\frac{K(\Sigma G)}{\langle X_1 - Y_1, \dots, X_d - Y_d\rangle} \cong \frac{k[X_1, \dots, X_d]}{(I(G) + \langle X_1^2, \dots, Y_d^2\rangle)}$.

4.2.1 Primary decomposition

Definition 4.18. Let $I \leqslant R$. $\mathfrak{p} \in \operatorname{Spec}(R)$ is called a *minimal prime ideal of* I if $I \subseteq \mathfrak{p}$ and there is no $\mathfrak{p}' \in \operatorname{Spec}(R)$ such that $I \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}$.

We denote the set of minimal prime ideals of I by $\operatorname{Min}(I)$.

Theorem 4.19. *Let $I \leqslant_m R$. Then $I = \bigcap_{i=1}^n \underline{X}^{\underline{a}_i}$ for some $n \geqslant 1$ and $\underline{a}_1, \dots, \underline{a}_n \subseteq \mathbb{N}_0^d$.*

Theorem 4.20. *Let $I \leqslant_m R$ be square-free. Then $I = \bigcap_{i=1}^n \underline{X}^{\underline{a}_i}$ for some $n \geqslant 1$ and $\underline{a}_1, \dots, \underline{a}_n \subseteq \{0,1\}^d$.*

Lemma 4.21. If $I \leqslant R$ has an irredundant decomposition $I = \bigcap_{i=1}^m \mathfrak{p}_i$ as an intersection of prime ideals, then $\operatorname{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, where $\mathfrak{p} \leqslant_m R$.

Corollary 4.22. Let $I \subseteq S$ be a square-free monomial ideal. Then $I = \bigcap_{\mathfrak{p} \in \operatorname{Min}(I)} \mathfrak{p}$, where $\mathfrak{p} \leqslant_m R$.

Definition 4.23. The *support* of M is

$$\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

Definition 4.24. $\mathfrak{p} \in \operatorname{Spec}(R)$ is called a *minimal prime ideal* of M if $M_{\mathfrak{p}} \neq 0$ and for $\mathfrak{p} \supseteq \mathfrak{p}'$ with $\mathfrak{p}' \in \operatorname{Spec}(R)$, one has $M_{\mathfrak{p}'} = 0$, i.e., $\mathfrak{p} \in \operatorname{Supp}_R(M)$ and for $\mathfrak{p} \supseteq \mathfrak{p}' \in \operatorname{Spec}(R)$, $\mathfrak{p}' \notin \operatorname{Supp}_R(M)$.

Remark. Note $(R/I)_{\mathfrak{p}} \neq 0$ if and only if $I_{\mathfrak{p}} \lneq R_{\mathfrak{p}}$ if and only if $I \cap R \smallsetminus \mathfrak{p} = \emptyset$ if and only if $I \subseteq \mathfrak{p}$, and $(R/I)_{\mathfrak{p}'} = 0$ if and only if $I \nsubseteq \mathfrak{p}'$, similarly.

So $\mathfrak{p} \in \operatorname{Supp}_R(R/I)$ if and only if $I \subseteq \mathfrak{p} \in \operatorname{Spec}(R)$ if and only if $\mathfrak{p} \in \operatorname{V}(I)$. Thus, $\operatorname{Supp}_R(R/I) = \operatorname{V}(I)$.

Also, $\mathfrak{p} \in \operatorname{Spec}(R)$ is a minimal prime ideal of R/I if and only if $I \subseteq \mathfrak{p}$ and there is no $\mathfrak{p}' \in \operatorname{Spec}(R)$ such that $I \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}$. Thus, $\operatorname{Min}(\operatorname{Supp}_R(R/I)) = \operatorname{Min}(I) = \operatorname{Min}(R/I)$.

Corollary 4.25. Let $I \leqslant_m R$. Then $\mathfrak{p} \leqslant_m R$ for $\mathfrak{p} \in \operatorname{Ass}_R(I)$.

Corollary 4.26. Let $I \leqslant_m R$ and $\mathfrak{p} \in \operatorname{Ass}_R(I)$. Then there exists $h \in [\![R]\!]$ such that $\mathfrak{p} = (I : h)$.

Proof. Since R is noetherian and $\mathfrak{p} \in \operatorname{Ass}_R(I)$, there exists $f \in R$ such that $\mathfrak{p} = (I : f)$. Since $I \leqslant_m R$, we have $g \in (I : f)$ if and only if $gf \in I$ if and only if $gu \in I$ for all u in monomials of f if and only if $g \in \bigcap_{u \text{ monomials of } f} (I : u)$. So $\mathfrak{p} = (I : f) = \bigcap_{u \text{ monomial of } f} (I : u)$. Thus, $\mathfrak{p} = (I : u)$ for some $u \in [\![R]\!]$. $\square$

Proposition 4.27. Let $I \leqslant R$ and $S = R/I$. Then there exists a polynomial ring R' and a square-free monomial ideal I' such that $S = S'/(\underline{\alpha})$, where $S' = R'/I'$ and $\underline{\alpha}$ is a regular sequence on S' of forms of degree 1.

Proof. Let $F = \{f_1, \ldots, f_r\}$ be a set of monomials that minimally generate I. Assume without loss of generality X_1 occurs in at least one of the monomials in F with multiplicity greater than 1, say f_1 Then one may write $f_1 = X_1^{a_1} g_1, \ldots, f_s = X_1^{a_s} g_s$, where $a_1 \geqslant 2$, $a_2, \ldots, a_s \geqslant 1$, $X_1 \nmid g_i$ for $i = 1, \ldots, s$ and $X_1 \nmid f_i$ for $i = s+1, \ldots, r$. Set

$$I' = (X_0 X_1^{a_1 - 1} g_1, \ldots, X_0 X_1^{a_s - 1} g_s, f_{s+1}, \ldots, f_r) \subseteq R' = R[X_0],$$

where X_0 is a new variable. Claim. $X_0 - X_1$ is a nonzero divisor of $S' = \frac{R'}{I'}$. Suppose not, then $X_0 - X_1 \in \operatorname{ZD}(R'/I') = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(R'/I')} \mathfrak{p}$, so $X_0 - X_1 \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R(R'/I') = \operatorname{Ass}_R(I')$ since R' is noetherian. By previous corllary, we have $\mathfrak{p} \leqslant_m R'$, so $X_0, X_1 \in \mathfrak{p}$. Also, by previous corollary, $\mathfrak{p} = (I' : h)$ for some $h \in [\![R]\!]$. So $X_0 h, X_1 h \in I'$. Also, since $h \notin I'$, through the restricted form of genertors of I, we must have $X_1 h = X_0 X_1^{a_i - 1} g_i h_1$ for some $i \in \{1, \ldots, s\}$ and $h_1 \in R'$. Since $X_0 h \in I'$, we have

$$X_0^2 X_1^{a_1 - 2} g_i h_1 = X_0 h = \begin{cases} X_0 X_1^{a_j - 1} M & \text{for some } j \in \{1, \ldots, s\} \text{ and } M \in R' \text{ or} \\ f_j M & \text{for some } j \in \{s+1, \ldots, r\} \text{ and } M \in R' \end{cases}.$$

Since X_0 is a new variable, $X_0 \nmid f_j$. So in bothe cases, $X_0 \mid M$. Then $h = \begin{cases} X_1^{a_j - 1} M & \text{or} \\ f_j \frac{M}{X_0} \end{cases} \in I'$, a contradiction. Thus, $X_0 - X_1$ is regular for $R'/I' = S'$. Since

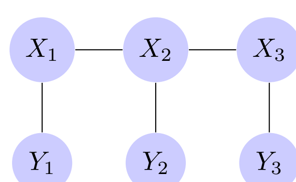
$$\frac{S'}{(X_0 - X_1)} \cong \frac{R'}{(I' + (X_0 - X_1))} \cong \frac{R}{(X_1^{a_1} g_1, \ldots, X_1^{a_s} g_s, f_{s+1}, \ldots, f_r)} = \frac{R}{(f_1, \ldots, f_r)} = \frac{R}{I} = S,$$

one can repeat the contruction to obtain the asserted monomial ideal I'. $\square$

Remark. The ideal I' constructed above is called the *polarization* of I. Thus, any monomial ring is a deformation by linear forms of a monomial ring with square-free relations.

Note that I is Cohen-Macaulay (resp. Gorenstein) if and only if I' is Cohen-Macaulay (resp. Gorenstein).

Example 4.28. Let $R = A[X_1, X_2, X_3]$. Then $\frac{A[X_1, X_2, X_3]}{(X_1^1, X_2^2, X_3^2)} \cong \frac{k[X_1, X_2, X_3, Y_1, Y_2, Y_3]}{(X_1 Y_1, X_2 Y_2, X_3 Y_3) + (X_1 - Y_1, X_2 - Y_2, X_3 - Y_3)}$.



4.3 Graphs and edge ideals

Graph means here finite simple undirected graph. We will take the more combinatorial approach (as oppose to the geometric approach) to the study of graphs. However, these objects still model important objects from other area, like social networks and electrical power systems.

Definition 4.29. For $n \geqslant 1$, let P_n be a path on n vertices, i.e.,

$$v_1 \text{ --- } v_2 \text{ --- } \cdots \text{ --- } v_n$$

Definition 4.30. Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$. The *edge ideal* of G is defined by

$$I_G = (\{X_i X_j \mid v_i v_j \text{ is an edge in } G\}).$$

Remark. By definition, the edge I_G is square-free.

4.4 Decomposition of edge ideal

Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$.

Definition 4.31. A *vertex cover* of G is a subset of $V' \subseteq V$ such that for any $v_i v_j \in E$, either $v_i \in V'$ or $v_j \in V'$.

A vertex cover V' is *minimal* if it doesn't properly contain another vertex cover.

Fact 4.32. (a) $\{$vertex cover of $G\}$ is closed under supersets.

(b) If $|V| < \infty$, then every vertex cover contains a minimal one.

(c) V itself is a vertex cover for finite graph G, so there exists a minimal vertex cover.

Lemma 4.33. Let $V' \subseteq V$. Then $I(G) \subseteq P_{V'}$ if and only if V' is a vertex cover of G.

Proof. Let $v_i v_j \in E$ be arbitrary. Then $I(G) \subseteq P_{V'}$ if and only if $X_i X_j \in P_{V'}$ if and only if $X_k \mid X_i X_j$ for some $v_k \in V'$ if and only if $X_k = X_i$ or $X_k = X_j$ for some $v_k \in V'$ if and only if $v_i = v_k \in V'$ or $v_j = v_k \in V'$ for some $v_k \in V'$ if and only if V' is a vertex cover of G. ☐

Theorem 4.34. $I(G) = \bigcap_{V' \text{ v. cover}} P_{V'} = \bigcap_{V' \text{ min. v. cover}} P_{V'}$. *These are m-irreducible decompositions and the second decomposition is irredundant.*

Proof. Since $\{V' \text{ v. cover}\} \supseteq \{V' \text{ min. v. cover}\}$, $\bigcap_{V' \text{ v. cover}} P_{V'} \subseteq \bigcap_{V' \text{ min. v. cover}} P_{V'}$. Let $\alpha \in \bigcap_{V' \text{ min. v. cover}} P_{V'}$. Let V' be a vertex cover for G. Then there exists $V'' \subseteq V'$ such that V'' is a minimal vertex cover. So $\alpha \in P_{V''} \subseteq P_{V'}$. Hence $\alpha \in \bigcap_{V' \text{ v. cover}} P_{V'}$. Thus, $\bigcap_{V' \text{ v. cover}} P_{V'} \subseteq \bigcap_{V' \text{ min. v. cover}} P_{V'}$.

By previous lemma, $I(G) \subseteq P_{V'}$ for any vertex cover V'. So $I(G) \subseteq \bigcap_{V' \text{ v. cover}} P_{V'}$. Since $I(G)$ is square-free, by previous theorem, there are $V_1, \ldots, V_n \subseteq V$ such that $I(G) = \bigcap_{i=1}^n P_{V_i} \subseteq P_{V_k}$ for $k = 1, \ldots, n$. So V_k is a vertex cover for $k = 1, \ldots, n$ by previous lemma. Then $\bigcap_{V' \text{ v. cover}} P_{V'} \subseteq \bigcap_{i=1}^n P_{V_i} = I(G)$.

Besides, let $V', V'' \subseteq V$ be satisfying $V' \nsubseteq V'' \nsubseteq V'$. Then $P_{V'} \nsubseteq P_{V''} \nsubseteq P_{V'}$. So the second decomposition is irredundant. ☐

Remark. This can be used to decompose any square-free quadratic monomial ideal.

Definition 4.35. Specify a length l. The *path ideal* is $I_l(G) = \langle$generated by the P_l's in $G\rangle$.

Remark. $I_2(G) = I(G)$.

Definition 4.36. An *l-vertex cover* is a subset $V' \subseteq V$ such that for any $P_l : v_{i_1}, \dots, v_{i_l}$, we have $v_{i_j} \in V'$ for some j.

Lemma 4.37. Let $V' \subseteq V$. Then V' is an l-vertex cover if and only if $I_l(G) \subseteq P_{V'}$.

Let $\Gamma \subseteq G$ be a fixed graph.

Definition 4.38. Define

$$I_\Gamma(G) = \langle \text{all isomorphic copies of } \Gamma \subseteq G \rangle.$$

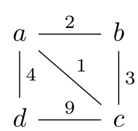
Definition 4.39. Γ-*vertex cover*: $V' \subseteq V$ such that for any isomorphic copy T of $\Gamma \subseteq G$, some vertex in T is in V'.

Lemma 4.40. Let $V' \subseteq V$. Then V' is a Γ-vertex cover if and only if $I_\Gamma(G) \subseteq P_{V'}$.

Theorem 4.41. *$I_\Gamma(G) = \bigcap_{V' \text{: } \Gamma\text{-v. cover}} P_{V'} = \bigcap_{V' \text{: min. } \Gamma\text{-v. cover}} P_{V'}$. The second decomposition is irredundant.*

Example 4.42. Let $I_3(G) = \langle abc, abd, abf, ade, adf \rangle$. Then $\langle a \rangle \cap \langle b, e, f \rangle \cap \langle b, d \rangle \cap \langle c, d, f \rangle = \langle abc, abd, abf, ade \rangle$.

Definition 4.43. Weighted graph:



Define

$$I_w(G) = \langle a^2b^2, ac, a^4d^4, b^3c^3, c^9d^9 \rangle.$$