Boundary Control Method

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Contents

Chapter 1

Partial Dififerention equation

1.1 Tranport Equation

Let's consider the 1D transport equation, which is the following initial value problem (IVP)

$$\begin{cases} u_t(x,t) + bu_x(x,t) = 0, & x \in \mathbb{R}, t \in \mathbb{R}^{>0}, \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $b \in \mathbb{R}$ is a constant and $g \in \mathcal{C}^1(\mathbb{R})$.

To solve it, we try to reduce the problem to an ODE along some curve $x = x(t)$. Consider the curve defined by $\frac{dx}{dt} = b$, then $x(t) = bt + x_0$, where $x_0 = x(0)$. By the chain rule, along such curve we have

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t \frac{dt}{dt} = bu_x + u_t = 0.$$

This implies $u(x(t), t) = \text{const} = u(x_0, 0) = g(x_0) = g(x(t) - bt)$. As the above lines $x = bt + x_0$ (called *characteristic lines*) fill out the entire xt-plane with some $x_0 \in \mathbb{R}$, we find the solution of the above IVP as $u(x,t) = g(x - bt)$. Indeed, as $g \in \mathcal{C}^1(\mathbb{R})$, we can check that $u(x,t) = g(x - bt)$ satisfies the PDE

$$u_t(x,t) + bu_x(x,t) = g'(x - bt)(-b) + bg'(x - bt) = 0, x \in \mathbb{R}, t \in \mathbb{R}^{>0}$$

and the initial condition $u(x, 0) = g(x - b \cdot 0) = g(x), x \in \mathbb{R}$.

Remark. As the solution indicates, the solution corresponds to "transport" the initial data $g(x)$ along the x-axis at a constant speed b.

Now we consider the general case of the transport equation in $\mathbb{R}^n$ and explore the similar idea to solve it from a different perspective. Consider the following IVP

$$\begin{cases} u_t(\boldsymbol{x},t) + \boldsymbol{b} \cdot \nabla u(\boldsymbol{x},t) = 0, & \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\ u(\boldsymbol{x},0) = g(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^n, \end{cases}$$

where $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ is a constant vector and $g \in \mathcal{C}^1(\mathbb{R}^n)$.

To solve the above IVP, we notice first that the equation can be written as

$$u_t + \boldsymbol{b} \cdot \nabla u = \langle \boldsymbol{b}, 1 \rangle \cdot \langle \nabla u, u_t \rangle = 0,$$

which implies the directional derivative of $u(\boldsymbol{x}, t)$ along the direction $(\boldsymbol{b}, 1)$ is 0. So $u(\boldsymbol{x}, t)$ must be constant along this direction. To see this, we let

$$z(s) := u(\boldsymbol{x} + s\boldsymbol{b}, t + s), s \in \mathbb{R}.$$

Then differentiating z with respect to s, we get

$$\begin{aligned} z'(s) = \frac{dz(s)}{ds} &= \nabla u(\boldsymbol{x} + s\boldsymbol{b}, t + s) \cdot \frac{d(\boldsymbol{x} + s\boldsymbol{b})}{ds} + \frac{\partial}{\partial t} u(\boldsymbol{x} + s\boldsymbol{b}, t + s) \frac{d(t + s)}{ds} \\ &= \nabla u(\boldsymbol{x} + s\boldsymbol{b}, t + s) \cdot \boldsymbol{b} + u_t(\boldsymbol{x} + s\boldsymbol{b}, t + s) = 0. \end{aligned}$$

So $z(s)$ is a contant in s. In particular, $z(0) = z(-t)$, which implies

$$u(\boldsymbol{x}, t) = u(\boldsymbol{x} - t\boldsymbol{b}, 0) = g(\boldsymbol{x} - t\boldsymbol{b}).$$

Again, we can easily check the above solution indeeds solves the above IVP.

Next, let us look at the nonhomogeneous problem

$$\begin{cases} u_t(\boldsymbol{x}, t) + \boldsymbol{b} \cdot \nabla u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), & \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\ u(\boldsymbol{x}, 0) = g(\boldsymbol{x}), & \boldsymbol{x} \in \mathbb{R}^n, \end{cases}$$

where again $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ is a constant vector and $g \in \mathcal{C}^1(\mathbb{R}^n)$ and $f \in \mathcal{C}^1(\mathbb{R}^n \times (0, \infty))$.

To solve the above IVP, we follow the above example and let $z(s) = u(\boldsymbol{x} + s\boldsymbol{b}, t + s)$ and then we have

$$z'(s) = \nabla u(\boldsymbol{x} + s \cdot \boldsymbol{b}, t + s) \cdot \boldsymbol{b} + u_t(\boldsymbol{x} + s\boldsymbol{b}, t + s) = f(\boldsymbol{x} + s\boldsymbol{b}, t + s).$$

Hence

$$u(\boldsymbol{x}, t) - u(\boldsymbol{x} - t\boldsymbol{b}, 0) = z(0) - z(-t) = \int_{-t}^{0} z'(s) ds = \int_{-t}^{0} f(\boldsymbol{x} + s\boldsymbol{b}, t + s) ds = \int_{0}^{t} f(\boldsymbol{x} + (s - t)\boldsymbol{b}, s) ds,$$

i.e.,

$$u(\boldsymbol{x}, t) = g(\boldsymbol{x} - t\boldsymbol{b}) + \int_0^t f(\boldsymbol{x} + (s - t)\boldsymbol{b}, s) ds.$$

To check the above $u(\boldsymbol{x}, t)$ is indeed a solution of the above IVP, as $g \in \mathcal{C}^1(\mathbb{R}^n)$ and $f \in \mathcal{C}^1(\mathbb{R}^n \times (0, \infty))$, by the lemma below, we compute

$$u_t(\boldsymbol{x}, t) = \nabla g(\boldsymbol{x} - t\boldsymbol{b}) \cdot (-\boldsymbol{b}) + f(\boldsymbol{x}, t) + \int_0^t \nabla f(x + (s - t)\boldsymbol{b}, s) \cdot (-\boldsymbol{b}) ds,$$

and

$$\nabla u(\boldsymbol{x}, t) = \nabla g(\boldsymbol{x} - t\boldsymbol{b}) + \int_0^t \nabla f(\boldsymbol{x} + (s - t)\boldsymbol{b}, s) ds.$$

Combine the above 2 equations, we really get $u_t(\boldsymbol{x}, t) + \boldsymbol{b} \nabla u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t)$. In addition, the initial condition $u(\boldsymbol{x}, 0) = g(\boldsymbol{x}) + \int_0^0 \nabla f(\boldsymbol{x} + s\boldsymbol{b}, s) ds = g(\boldsymbol{x})$ is satisfied.

Lemma 1.1. Let $F \in \mathcal{C}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\int_0^t F(t,s)ds = F(t,t) + \int_0^t F_t(t,s)ds.$$

Proof. Since $F \in \mathcal{C}^1(\mathbb{R}^2)$, by DCT and MVT,

$$\begin{aligned}
\frac{d}{dt}\int_0^t F(t,s)ds &= \lim_{h\to 0}\frac{1}{h}\left[\int_0^{t+h} F(t+h,s)ds - \int_0^t F(t,s)ds\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int_0^{t} (F(t+h,s) - F(t,s))ds + \int_t^{t+h} F(t+h,s)ds\right] \\
&= \lim_{h\to 0}\frac{1}{h}\int_0^{t} (F(t+h,s) - F(t,s))ds + \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} F(t+h,s)ds \\
&= \int_0^t \lim_{h\to 0}\frac{1}{h}(F(t+h,s) - F(t,s))ds + \lim_{h\to 0}\frac{1}{h}F(t+h,\xi)(t+h-t), \xi \in [t,t+h] \\
&= \int_0^t F_t(t,s)ds + F(t,t).
\end{aligned}$$

□

1.2 Wave Equation in One Dimension

Let's consider first the 1D wave equation that models small vibrations of a string:

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0,\ \ x \in \mathbb{R}, t \in \mathbb{R}^{>0},$$

where $c > 0$ is a constant (propagation speed). The equation is of hyperbolic type and its characteristics are given by $x \pm ct = \text{const}$. Then by a change of variable $\mu(x,t) = x + ct$ and $\eta(x,t) = x - ct$, the above wave equation is transformed to $u_{\mu\eta} = 0$ which has general solutions $u(\mu,\eta) = F(\mu) + G(\eta)$ with arbitrary $F, G \in \mathcal{C}^1(\mathbb{R})$. Return to the variables x and t, we have the general solution of the above wave equation given by

$$u(x,t) = F(x+ct) + G(x-ct).$$

Note that the solution is a sum of solutions of two transport equations and it can be realized as the superposition of two waves propagating with constant speed c in the opposite directions along the x-axis.

If $F, G \in \mathcal{C}^2(\mathbb{R})$, then $u(x,t) = F(x+ct) + G(x-ct)$ becomes a strong solution. We can also use the following algebraic property of the solution u to define a weak solution. To see this, consider a rectangle $ABCD$ in the $\mu\eta$-plane whose sides are parallel to the coordinates axes. Since F has a constant value along the vertical lines and G has a constant value along horizontal lines, we have

$$u(A) + u(C) = u(B) + u(D),$$

where $A, B, C, D \in \mathbb{R}^2$. Translated to the xy-plane, we may see above equality as a parallelogram rule that holds for every parallelogram whose sides are all segments of the characteristics.

Definition 1.2. If a function $u(x,t)$ satisfies $u(A)+u(C)=u(B)+u(D)$ for every parallelogram $ABCD$ whose sides are all segments of the characteristics of the above wave equation is called a *weak* solution.

Remark. The characterization of weak solutions is useful for solving the wave equation with both initial and boundary conditions.

Let's consider the Cauchy problem for the above wave equation as an initial value problem (IVP), that is

$$\begin{cases} u_{tt}(x,t)-c^2u_{xx}(x,t)=0, & x\in\mathbb{R}, t\in\mathbb{R}^{>0}, \\ u(x,0)=g(x),\ u_t(x,0)=h(x), & x\in\mathbb{R}, \end{cases}$$

where g and h are arbitrary functions. Plug in the initial conditions into the general solution $u(x,t)=F(x+ct)+G(x-ct)$, we get

$$\begin{cases} F(x)+G(x)=u(x,0)=g(x) \\ cF'(x)-cG'(x)=u_t(x,0)=h(x). \end{cases}$$

Integrate the second equation we get $F(x)-G(x)=\frac{1}{c}\int_0^x h(y)dy+[F(0)-G(0)]$. Combine with the first equation we have

$$\begin{cases} F(x)=\frac{1}{2}g(x)+\frac{1}{2c}\int_0^x h(y)dy+\frac{1}{2}[F(0)-G(0)] \\ G(x)=\frac{1}{2}g(x)-\frac{1}{2c}\int_0^x h(y)dy-\frac{1}{2}[F(0)-G(0)]. \end{cases}$$

Thus, we get the solution of the above IVP (called the d'Alembert formula)

$$u(x,t)=\frac{1}{2}[g(x+ct)+g(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}h(y)dy.$$

Alternative: in view of the form of the general solution, the above IVP can also be solved by the following way that involves solving two 1D transport equations. To see this, note $u_{tt}-c^2u_{xx}=0$ can be "factored" as

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)u=0.$$

Let $v(x,t)=(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x})u=u_t-cu_x$, then v satisfies the transport equation with initial condition

$$\begin{cases} v_t(x,t)+cv_x(x,t)=0 \\ v(x,0)=h(x)-cg'(x)r. \end{cases}$$

From previous section, the above equation has the solution

$$v(x,t)=v(x-ct)=h(x-ct)-cg'(x-ct).$$

Also we have u satisfies the nonhomogeneous transport equation

$$\begin{cases} u_t(x,t)-cu_x(x,t)=v(x,t) \\ u(x,0)=g(x) \end{cases}$$

Again, from previous section, this nonhomogeneous equation has the solution

$$\begin{aligned} u(x,t) &= g(x+ct) + \int_0^t v(x-c(s-t),s)ds \\ &= g(x+ct) + \int_0^t h(x+ct-2cs)ds - c\int_0^t g'(x+ct-2cs)ds \\ &= g(x+ct) - \frac{1}{2c}\int_{x+ct}^{x-ct} h(y)dy + \frac{1}{2}g(x+ct-2cs)\mid_{s=0}^{s=t} \\ &= \frac{1}{2}[g(x+ct)+g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy, \end{aligned}$$

which coincides with the d'Alembert formula.

Theorem 1.3. *Let $g \in \mathcal{C}^2(\mathbb{R})$ and $h \in \mathcal{C}^1(\mathbb{R})$. Then the IVP problem*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, & x \in \mathbb{R}, t \in \mathbb{R}^{>0}, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \mathbb{R}, \end{cases}$$

is well-posed, and its solution is given by the d'Alembert formula

$$u(x,t) = \frac{1}{2}[g(x+ct)+g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy.$$

Moreover, the solution is a strong solution, i.e., $u \in \mathcal{C}^2(\mathbb{R}\times(0,\infty))$.

Proof. Exercise. □

Note from the d'Alembert formula, the solution u of the IVP at (x,t) is only determined by the initial position at $x \pm ct$ and the initial velocity along the segment from $x-ct$ to $x+ct$, nothing outside the interval $[x-ct, x+ct]$ matters. We call the closed interval $[x-ct, x+ct]$ the *domain of dependence* of (x,t). On the other hand, the initial data at any point x_0 on the initial time line $t=0$ must influence all values $u(x,t)$ in the wedge formed by two characteristics drawn from x_0 into the region of $t>0$. This wedge area, $\{(x,t) \mid t>0, x_0-ct \leqslant x \leqslant x_0+ct\}$ is called the *range of influence* of x_0. Similarly, we may also define the range of influence of an interval $[x_1,x_2]$ on the initial time line $t=0$ as $\{(x,t) \mid t>0, x_1-ct \leqslant x \leqslant x_2+ct\}$.

Next, we consider the nonhomogeneous wave equation

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), & x \in \mathbb{R}, t \in \mathbb{R}^{>0}, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \mathbb{R}, \end{cases}$$

where g and h are arbitrary functions. To solve the homogeneous problem, we first decompose the problem by linearity as $u = u_1 + u_2$, where u_1 solves the nonhomogeneous problem and u_2 solves the nonhomogeneous problem however with zero initial data. Namely, we have

$$\begin{cases} u_{1_{tt}}(x,t) - c^2 u_{1_{xx}}(x,t) = 0 \\ u_1(x,0) = g(x),\ u_{1_t}(x,0) = h(x). \end{cases} \qquad \begin{cases} u_{2_{tt}}(x,t) - c^2 u_{2_{xx}}(x,t) = f(x,t) \\ u_2(x,0) = 0 = u_{2_t}(x,0). \end{cases}$$

To find u_1, we simply apply the d'Alembert formula, so the problem becomes to find the solution u_2. A general way to solve nonhomogeneous problems like the u_2 problem with zero initial conditions is to use the Duhamel's Principle, which we state in the following.

Theorem 1.4 (Duhamel's Principal)**.** *If $w(x,t;s)$ ($s > 0$ is a parameter) solves the homogeneous problem*

$$\begin{cases} w_{tt}(x,t;s) - c^2 w_{xx}(x,t;s) = 0, & x \in \mathbb{R}, t > s \\ w(x,s;s) = 0,\ w_t(x,s;s) = f(x,s), & x \in \mathbb{R}. \end{cases}$$

Then the function $u(x,t) = \int_0^t w(x,t;s)ds$ solves the u_2-problem.

Proof. We check by a direct computation, using the initial conditions for w and previous lemma,

$$\begin{aligned} u_t(x,t) &= w(x,t;t) + \int_0^t w_t(x,t;s)ds = \int_0^t w_t(x,t;s)ds \\ u_{tt}(x,t) &= w_t(x,t;t) + \int_0^t w_{tt}(x,t;s)ds = f(x,t) + \int_0^t w_{tt}(x,t;s)ds \\ u_{xx}(x,t) &= \int_0^t w_{xx}(x,t;s)ds. \end{aligned}$$

Then we get

$$u_{tt} - c^2 u_{xx} = f(x,t) + \int_0^t (w_{tt}(x,t;s) - c^2 w_{xx}(x,t;s))ds = f(x,t).$$

In addition, it is obvious that the initial conditions in the u_2-problem are satisfied. □

Thus, to find u_2 we only need to find the solution $w(s,t;s)$ of

$$\begin{cases} w_{tt}(x,t;s) - c^2 w_{xx}(x,t;s) = 0, & x \in \mathbb{R}, t > s \\ w(x,s;s) = 0,\ w_t(x,s;s) = f(x,s), & x \in \mathbb{R}. \end{cases}$$

and then integrate the parameter s from 0 to t. Notice the above equation is a homogeneous wave equation starting at time $t = s$, letting $\tilde{t} = t - s$, we may apply d'Alembert formula to find

$$w(x,\tilde{t};s) = \frac{1}{2c}\int_{x-c\tilde{t}}^{x+c\tilde{t}} f(y,s)dy,$$

which implies

$$w(s,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy.$$

Therefore, by Duhamel's principle the solution of the u_2-problem is given by

$$u_2(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds = \frac{1}{2c}\iint_{\mathcal{D}} f(y,s)dyds,$$

where $\mathcal{D}$ is the triangular region $\{(y,s) \mid 0 < s < t, x - c(t-s) \leqslant y \leqslant x + c(t-s)\}$ in the ys-plane. To summarize, we have the following result for the nonhomogeneous problem.

Theorem 1.5. *Let f be $\mathcal{C}^1$ in x and continuous in t, $g \in \mathcal{C}^2(\mathbb{R})$, $h \in \mathcal{C}^1(\mathbb{R})$. Then the nonhomogeneous problem*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), & x \in \mathbb{R}, t \in \mathbb{R}^{>0}, \\ u(x,0) = g(x),\ u_t(x,0) = h(x), & x \in \mathbb{R}, \end{cases}$$

is well-posed, and its solution is given by the formula

$$u(x,t) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds.$$

Moreover, the solution is a strong solution, i.e., $u \in \mathcal{C}^2(\mathbb{R}\times(0,\infty))$.

As another application of the d'Alembert formula, next we consider a half-space problem for the wave equation, that is, wave equation on the half real number line $\mathbb{R}^{>0}$. More precisely, we consider

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & x \in \mathbb{R}^{>0}, t \in \mathbb{R}^{>0}, \\ u(x,0) = g(x), u_t(x,0) = h(x), & x \in \mathbb{R}^{>0}, \\ u(0,t) = 0, & t \in \mathbb{R}^{>0}, \end{cases}$$

where the last condition $u(0,t) = 0$ is called the homogeneous Dirichlet boundary condition. Physically this means the end of the string at $x = 0$ is held fixed at all time.

The idea to solve the half-space problem is to extend the data to the entire real number line so that we may apply the d'Alembert formula. Namely, we assume that there is still string on $\mathbb{R}^{<0}$ and it is just during the vibration the point $x = 0$ is kept fixed. Suppose we extend g, h, u to the entire real numebr line, and we use $\tilde{g}, \tilde{h}, \tilde{u}$ to represent the new functions. Then from the d'Alembert formula, we should have

$$\tilde{u}(x,t) = \frac{1}{2}[\tilde{g}(x+ct) + \tilde{g}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(y)dy.$$

Plug in the boundary condition $\tilde{u}(0,t) = 0$ we then get

$$0 = \frac{1}{2}[\tilde{g}(ct) + \tilde{g}(-ct)] + \frac{1}{2c}\int_{-ct}^{ct} \tilde{h}(y)dy$$

which will always hold if $\tilde{g}, \tilde{h}$ are both odd functions. So we only need to extend g, h to $R^{<0}$ by an odd relection

$$\tilde{g}(x) = \begin{cases} g(x), & x \geqslant 0, \\ -g(-x), & x < 0, \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x), & x \geqslant 0, \\ -h(-x), & x < 0. \end{cases}$$

Put these functions into the d'Alembert formula, we find the solution of the half-space problem as

$$u(x,t) = \begin{cases} \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy, & x \geqslant ct \geqslant 0 \\ \frac{1}{2}[g(x+ct) - g(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} h(y)dy, & 0 \leqslant x \leqslant ct. \end{cases}$$

We can check that

$$u(x,t) = -u(-x,t), \forall (x,t) \in \mathbb{R}\times\mathbb{R}^+.$$

Remark. Even if we have $g \in \mathcal{C}^2(\mathbb{R}^{>0})$ and $h \in \mathcal{C}^1(\mathbb{R}^{>0})$, in general u above may not be a strong solution, i.e., $u \notin \mathcal{C}^2(\mathbb{R}^{>0}\times\mathbb{R}^{>0})$. To make u a strong solution, we need compatibility conditions. By directly computing derivatives from u above, we can see that u is continuous if we have $g(0) = 0$, u is in $\mathcal{C}^1$ if we have $g(0) = h(0) = 0$ and u is in $\mathcal{C}^2$ if we have $g(0) = h(0) = g''(0) = 0$.

Theorem 1.6 (Dirihlet Condition)**.** *Given $g \in \mathcal{C}^2[0,\infty), h \in \mathcal{C}^1[0,\infty)$ and $\phi \in \mathcal{C}^2(0,\infty)$, there is a unique $\mathcal{C}^2$ solution u of the homogeneous Cauchy initial value problem (IVP) of the wave equation,*

$$\begin{array}{ll} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, & (x,t) \in \mathbb{R}^{>0} \times \mathbb{R}^{>0}, \\ u(x,0) = g(x), & x \in \mathbb{R}^+, \\ u_t(x,0) = h(x), & x \in \mathbb{R}^+, \\ u(0,t) = \phi(t), & t \in \mathbb{R}^{>0}, \end{array}$$

where ϕ, g, h satisfies the compatibility condition $g(0) = \phi(0)$, $g''(0) = \phi''(0)$ and $h(0) = \phi'(0)$.

Proof. Define

$$\Omega_1 = \{(x,t) \in \mathbb{R}^{>0} \times \mathbb{R}^{>0} \mid x > ct\} \text{ and } \Omega_2 = \{(x,t) \in \mathbb{R}^{>0} \times \mathbb{R}^{>0} \mid x < ct\}.$$

For $(x,t) \in \Omega_1$, since the wave propagation speed is c, $\phi(t)$ does not affect Ω_1, we have the solution

$$u_1(x,t) = \frac{1}{2}[g(x-ct) + g(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy.$$

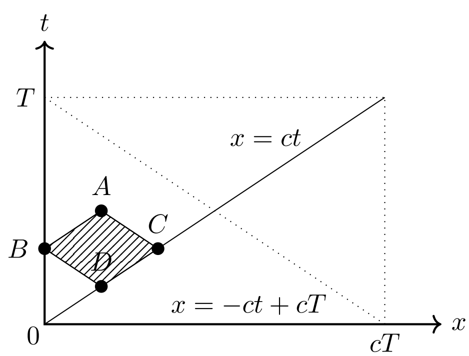
In particular, on the line $x = ct$, we get

$$\chi(x) := u_1(x, x/c) = \frac{1}{2}(g(0) + g(2x)) + \frac{1}{2c}\int_0^{2x} h(y)dy.$$

Let u_2 be the solution in Ω_2 of

$$\begin{array}{ll} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, & (x,t) \in \Omega_2, \\ u(x,x/c) = \chi(x), & x = ct, \\ u(0,t) = \phi(t), & t \in \mathbb{R}^{>0}. \end{array}$$

Fix $A := (x_0, t_0) \in \Omega_2$. One of the characteristic curve with slope $\frac{1}{c}$ through A intersects t-axis at $B := (0, t_0 - \frac{x_0}{c})$. The other characteristic curve with slope $-\frac{1}{c}$ through A intersects the line $ct = x$ at $C := \frac{1}{2}(ct_0 + x_0, t_0 + \frac{x_0}{c})$. The characteristic curve with slope $-\frac{1}{c}$ through B intersects $ct = x$ at $D := \frac{1}{2}(ct_0 - x_0, t_0 - \frac{x_0}{c})$.



The four points A, B, C, D form a parallelogram in Ω_2. By the parallelogram rule, $u(A) + u(D) = u(B) + u(C)$, i.e.,

$$u_2(x_0,t_0) + u_2\left(\frac{1}{2}\left(ct_0 - x_0, t_0 - \frac{x_0}{c}\right)\right) = u_2\left(0, t_0 - \frac{x_0}{c}\right) + u_2\left(\frac{1}{2}\left(ct_0 + x_0, t_0 + \frac{x_0}{c}\right)\right).$$

Hence for any $(x,t) \in \Omega_2$,

$$
\begin{aligned}
u_2(x,t) &= \phi(t - x/c) + \chi(1/2(ct+x)) - \chi(1/2(ct-x)) \\
&= \phi(t - x/c) + \frac{1}{2}(g(0) + g(ct+x)) + \frac{1}{2c}\int_0^{ct+x} h(y)dy \\
&\quad - \frac{1}{2}(g(0) + g(ct-x)) - \frac{1}{2c}\int_0^{ct-x} h(y)dy \\
&= \phi(t - x/c) + \frac{1}{2}(g(ct+x) - g(ct-x)) + \frac{1}{2c}\int_{ct-x}^{ct+x} h(y)dy.
\end{aligned}
$$

By setting

$$
u(x,t) = \begin{cases} u_1(x,t) & (x,t) \in \Omega_1 \\ u_2(x,t) & (x,t) \in \Omega_2 \end{cases}.
$$

and the fact that all derivatives of u are continuous along $ct = x$ line due to the compatibility condition, u is a solution. □

Chapter 2

Boundary Control Method in 1D Dynamical Inverse Problems [1]

2.1 Introduction

2.1.1 About this chapter

The purpose of this chapter is to recover the coefficient $q = q(x)$ in the string equation $\rho u_{tt} - u_{xx} + qu = 0$ with $\rho = \text{const} > 0$ on the semi-axis $x > 0$ via the time-domain measurements at the enedpoints $x = 0$ of the string.

2.1.2 Comment, notation, convention

- All functions in the chapter are real. The following classes of functions are in use:

  (a) the space $\mathcal{C}[a,b]$ of continuous functions and the space $\mathcal{C}^k[a,b]$ of k times continuously differentiable functions for $k \in \mathbb{N}$;

  (b) a Hilbert space $L_2(a,b)$ of square summable functions provided with the standard inner product

  $$(y,v)_{L_2(a,b)} := \int_b^a y(s)v(s)ds$$

  and the norm $\|y\|_{L_2(a,b)} := (y,y)^{\frac{1}{2}}_{L_2(a,b)}$;

  (c) the Sobolev class $H^1[a,b]$ of differentiable functions y with derivatives $y' \in L_2(a,b)$.

- Sometimes, to distinguish the time intervals $\alpha < t < \beta$ from the space ones $a < x < b$, we denote the space intervals by $\Omega^{(a,b)} := (a,b)$ and put $\Omega^a := (0,a)$.

- **Convention** All functions depending on time $t \geqslant 0$ are assumed to be extended to $t < 0$ by zero.

2.2 Forward problem

2.2.1 Statement

We deal with an initial boundary value problem (IBVP) of the form

$$\begin{array}{ll} \rho u_{tt} - u_{xx} + qu = 0, & x > 0, 0 < t < T \\ u(x,0) = 0 = u_t(x,0), & x \geqslant 0 \\ u(0,t) = f(t), & 0 \leqslant t \leqslant T, \end{array}$$

where $q = q(x) \in \mathcal{C}[0,\infty)$, $f = f(t)$ is a boundary control that may not satisfies the compatibility condition $f(0) = 0 = f'(0)$, $u = u^f(x,t)$ is a solution, which is interpreted as a wave initiated by f. When $q \neq 0$, we refer it as a perturbed problem. When $q = 0$, it becomes an unperturbed problem and the solution is denoted by $\tilde{u}^f(x,t)$, which can be found explicitly:

$$\tilde{u}^f(x,t) = f(t - \frac{x}{c}),$$

where $c := \frac{1}{\sqrt{\rho}}$ is a wave velocity.

2.2.2 Integral equation and generalized solutions

The main tool for investigating the IBVP is an integral equation. Seeking for the solution in the form $u^f = \tilde{u}^f + w$ with a new unknow $w = w^f(x,t)$, we easily get

$$\begin{array}{ll} \rho w_{tt} - w_{xx} = -qw - q\tilde{u}^f, & x > 0, 0 < t < T \\ w(x,0) = 0 = w_t(x,0), & x \geqslant 0 \\ w(0,t) = 0, & 0 \leqslant t \leqslant T. \end{array}$$

Applying the D'Alembert formula, we arrive at the equation

$$w + Mw = -M\tilde{u}^f, \quad x > 0, 0 \leqslant t \leqslant T,$$

where an operator M acts by the rule

$$(Mw)(x,t) := \frac{1}{2c} \iint_{K_c(x,t)} q(\xi) w(\xi,\eta) d\xi d\eta,$$

and $K_c(x,t)$ is the trapezium bounded by the characteristic lines $t \pm \frac{x}{c} =$ const. We will see later that $u^f|_{\eta<\frac{\xi}{c}} = 0$, then $w^f(\xi,\eta)|_{\eta<\frac{\xi}{c}} = [u^f(\xi,\eta) - f(\eta - \frac{\xi}{c})]|_{\eta<\frac{\xi}{c}} = 0$. So the trapezium becomes a parallelogram.

The above equation is a second-kind Volterra type equation and it can be studied by the standard iteration method. As can be shown, if the control $f \in \mathcal{C}^2[0,T]$ satisfies $f(0) = f'(0) = f''(0) = 0$, then its solution w^f is in $\mathcal{C}^2$. As a result, the function $u^f = \tilde{u}^f + w$ turns out to be classical (strong) solution.

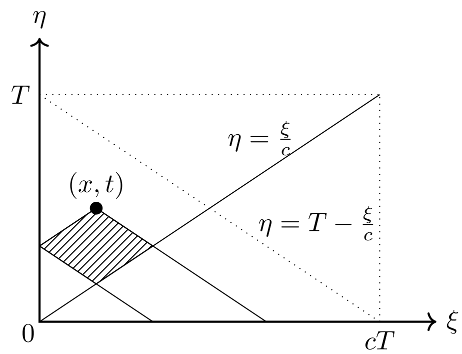


Figure 2.1: The domain $K_c(x,t)$

As is easy to see, for any $f \in L_2(0,T)$, $M\tilde{u}^f$ is a continuous function of x, t vanishing as $t < \frac{x}{c}$. Simple analysis shows that the equation $w + Mw = -M\tilde{u}^f, x > 0, 0 \leqslant t \leqslant T$ is uniquely solvable in this class of functions. In this case, we define the function $u^f := \tilde{u}^f + w^f$ to be the generalized (may be weak) solution.

2.2.3 Fundamental solution

Define

$$\delta_\epsilon := \begin{cases} \frac{1}{\epsilon}, & 0 \leqslant t < \epsilon, \\ 0, & t \geqslant \epsilon. \end{cases}$$

Recall that $\delta_\epsilon \to \delta$ as $\epsilon \to 0$ in the sense of distributions. Putting $f = \delta_\epsilon$ in $w + Mw = -M\tilde{u}^f, x > 0, 0 \leqslant t \leqslant T$, we get the solution w^{δ_ϵ} belonging to the class $\mathcal{C}([0,T], L_2(a,b))$. In the mean time, $M\tilde{u}^{\delta_\epsilon} \to M\tilde{u}^{\delta}$ as $\epsilon \to 0$, the latter being a continuously differentiable function of x, t for $0 \leqslant \frac{x}{c} \leqslant t \leqslant T$ and vanishing as $\frac{x}{c} > t$. Thus, even though the limit leads to a singular control $f = \delta$, the solution $w^\delta := \lim_{\epsilon \to 0} w^{\delta_\epsilon}$ does not leave the class $\mathcal{C}([0,T], L_2(a,b))$.
Simple analysis provides the following of its properties:

(a) w^δ is continuously differentiable in the domain $\{(x,t) \mid 0 \leqslant x \leqslant cT, \frac{x}{c} \leqslant t \leqslant T\}$ and $w^\delta|_{t<\frac{x}{c}} = 0$.

(b) $w^\delta(0,t) = 0$ for all $t > 0$.

(c) For any $x \geqslant 0$, the relation

$$w^\delta(x-0, \frac{x}{c}) = \lim_{s \uparrow x} w^\delta(s, \frac{x}{c}) = -\frac{c}{2}\int_0^x q(s)ds$$

holds and shows that w^δ may have a jump at the characteristic line $t = \frac{x}{c}$, whereas below this line, we have $w^\delta = 0$.

Proof. (3) Define a sequence of functions and distributions $\{\theta^j(t)\}_{j=-\infty}^{\infty}$, $-\infty < t < \infty$ by

$$\theta^0(t) := \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases} ; \theta^j(t) = \int_{-\infty}^{t} \theta^{j-1}(s)ds, j \in \mathbb{Z}.$$

This is said to be a smoothness scale: the bigger j, the smoother θ^j and

$$\theta^{j-1} = \frac{d\theta^j}{dt}(t), \quad -\infty < t < \infty$$

holds. In particular,

$$\theta^j(t) = \delta^{(-j-1)}(t), j \in \mathbb{Z}^{<0} \text{ and } \theta^j = \frac{t^j}{j!}\theta^0(t), j \in \mathbb{Z}^+.$$

Note that all elements of the scale vanish as $t < 0$.
Let us look for the fundamental solution in the form of a formal series

$$u^\delta(x,t) = \sum_{j=-1}^{\infty} a_j(x)\theta^j(t - \frac{x}{c})$$

with unknown functions a_j, which can be referred to as the Taylor expansion of u^δ near (from the left of) its forward front $t = \frac{x}{c}$. The reason to begin with the series with $j = -1$ is that in the unpertured case one has $\tilde{u}^\delta(x,t) = \delta(t - \frac{x}{c}) = \theta^{-1}(t - \frac{x}{c})$. Dif and only iferentiating and combining the similar terms, we easily get

$$\frac{1}{c^2}u^\delta_{tt} - u^\delta_{xx} + qu^\delta = \left[\frac{2}{c}a'_{-1}(x)\right]\theta^{-2}(t - \frac{x}{c}) +$$
$$\sum_{j=-1}^{\infty}\left[\frac{2}{c}a'_{j+1}(x) - a''_j(x) + q(x)a_j(x)\right]\theta^j(t - \frac{x}{c}) = 0.$$

By the independence of the different order singularities, we get the recurrent system of the ODEs: for $x > 0$,

$$\frac{2}{c}a'_{-1} = 0 \text{ and } \frac{2}{c}a'_j - a''_{j-1} + qa_{j-1} = 0,\ j \in \mathbb{Z}^+,$$

which is often called the transport equations and can be integrated one by one. In the same time, the condition $u^\delta(0,t) = \delta(t), t \geqslant 0$ implies the initial conditions

$$a'_{-1}(0) = 1 \text{ and } a'_j(0) = 0,\ j \in \mathbb{Z}^+.$$

The integration yields

$$a_{-1} = 1 \text{ and } a_0(x) = -\frac{c}{2}\int_0^x q(s)dx, \cdots.$$

Summarizing, we arrive at the well-known representation

$$u^\delta(x,t) = \delta(t - \frac{x}{c}) - \left[\frac{c}{2}\int_0^x q(s)ds\right]\theta^0(t - \frac{x}{c}) + \cdots$$

that describe the behavior of the fundamental solution near its forward front $t = \frac{x}{c}$.

□

In the unperturbed case of $q = 0$, we easily have $w^\delta = 0$, whereas

$$\tilde{u}^\delta = \delta(t - \frac{x}{c}),$$

satisfies the unperturbed equation in the sense of **distributions** and is called the fundamental solution of unperturbed problem.
Analogously,

$$u^{\delta}(x,t) = \delta(t - \frac{x}{c}) + w^{\delta}(x,t),$$

is said to be a fundamental solution of the perturbed problem. It describes the wave produced by the impluse control $f = \delta$.
Such a wave consists of the singular leading part $\delta(t - \frac{x}{c})$ propagating along the string with velocity c and the regular tail $w^{\delta}(x,t)$, which may have a jump at its **forward front**. The presence of the tail is explained by interaction between the singular part and the potential. Also, note that the singular part in the perturbed and unperturbed cases is one and the same.

2.2.4 Properties of waves

Return to the perturbed problem and represent the control in the form of a convolution with time: $f(t) = (\delta * f)(t) = \int_{-\infty}^{\infty} \delta(t-s)f(s)ds$. Since the potential q does not depend on t, we have

$$u^f = u^{\delta * f} = u^{\delta} * f = (\tilde{u}^{\delta} + w^{\delta}) * f = \tilde{u}^{\delta} * f + w^{\delta} * f,$$

which implies the representation

$$\begin{aligned} u^f(x,t) &= f(t - \frac{x}{c}) + \int_0^t w^{\delta}(x, t-s) f(s) ds, \\ &= f(t - \frac{x}{c}) + \int_0^{t-\frac{x}{c}} w^{\delta}(x,t-s)f(s)ds, \;\; x \geqslant 0, t \geqslant 0, \end{aligned}$$

which is often called the Duhamel's formula.
The listed below properties of the waves easily follow from the above equation:

(a) For any $f \in L_2(0,T)$, the relation

$$u^f|_{t<\frac{x}{c}} = 0, \quad x \geqslant 0, t \geqslant 0$$

hold, which is interpreted as the finiteness of the wave propagation speed.

(b) For $f \in L_2(0,T)$ and $\tau \in [0,T]$, define a delayed control $f_{\tau} \in L_2(0,T)$ by

$$f_{\tau}(t) := f(t-\tau), \quad 0 \leqslant t \leqslant T,$$

where τ is the value of delay. Since the potential does not depend on t, the relation

$$u^{f_{\tau}}(x,t) := u^f(x, t-\tau)$$

is valid.

(c) Consider in the upper triangular domain $\mathcal{D}$. Let $f(s), 0 \leqslant s \leqslant T$ be a **piece-wise continuous** function, which has a jump at $s = \xi$. Then $f(t - \frac{x}{c})$ has a jump at the characteristic line $t = \frac{c}{+}\xi$. Since $g(s) := w^{\delta}(x,s)$ is continuous on $s \in [\frac{x}{c}, t]$, $h(s) := w^{\delta}(x, t-s)$ is continuous on $s \in [0, t - \frac{x}{c}]$. Also, since $f(s)$ is **piece-wise continuous** on $s \in [0, t - \frac{x}{c}]$, we have $w^{\delta}(x,t-s)f(s)$ is piece-wise continuous on $s \in [0, t - \frac{x}{c}]$. So the integral has no jump at $t = \frac{x}{c} + \xi$. Thus, with a **fixed** $t \geqslant \xi \geqslant 0$, the integral term has no jump at the character line $t = \frac{x}{c} + \xi$. Then

$$\begin{aligned} u^f(c(t-\xi)+0,t) - u^f(c(t-\xi)-0,t) &= f\left(t - \frac{c(t-\xi)+0}{c}\right) - f\left(t - \frac{c(t-\xi)-0}{c}\right) \\ &= f(\xi - 0) - f(\xi + 0) = -\left(f(\xi+0) - f(\xi-0)\right), \end{aligned}$$

i.e.,

$$u^f(x,t)\mid_{x=c(t-\xi)-0}^{x=c(t-\xi)+0} = -f(s)\mid_{x=\xi-0}^{x=\xi+0}.$$

In particular, if $f(s)$ vanishes for $0 < s < T - \xi$ and has a jump at $s = T - \xi$, then $f(T - \xi - 0) = 0$. Since the action time is $T - (T - \xi) = \xi$, $u^f(\cdot, T) = 0$ when $x > c\xi$. So $u^f(c\xi + 0, T) = 0$. Replace t with T and ξ with $T - \xi$ in the previous expression, we have

$$-u^f(c\xi - 0, T) = u^f(x,t)\mid_{x=c\xi-0}^{x=c\xi+0} = -f(s)\mid_{x=T-\xi-0}^{x=T-\xi+0} = -f(T - \xi + 0),$$

i.e.,

$$u^f(c\xi - 0, T) = f(T - \xi + 0), \quad 0 \leqslant \xi \leqslant T.$$

2.2.5 Extended perturbed problem and locality

Return to $w + Mw = -M\tilde{u}^f, x > 0, 0 \leqslant t \leqslant T$ for the regular part w^δ of the fundamental solution. Representing in the form of the Neumann series $w^\delta = \sum_{k=0}^{\infty}(-1)^k M^k \tilde{u}^\delta$ and looking at the previous figure, we easily see that the values w^δ and then the values w^δ_x for $0 \leqslant x < \frac{cT}{2}, 0 < t < T - \frac{x}{c}$ are determined by the values of the potential $q|_{0<x<\frac{cT}{2}}$ only. Such a dependence is an inherence feature of the wave processes with the finite speed of the wave propagation; it is known as a locality principle. It motivates to extend the perturbed problem as follows:

$$\begin{array}{ll} \rho u_{tt} - u_{xx} + qu = 0, & (x,t) \in \Delta^{2T} \\ u|_{t<\frac{x}{c}} = 0 & \\ u(0,t) = f(t), & 0 \leqslant t \leqslant 2T, \end{array}$$

where $\Delta^{2T} := \{(x,t) \mid 0 < x < cT, 0 < t < 2T - \frac{x}{c}\}$. Seeking for the solution in the form $u^f = f(t - \frac{x}{c}) + w^f$, one can reduce the problem to the integral equation

$$w + Mw = -M\tilde{u}^f, \quad (x,t) \in \Delta^{2T},$$

which is quite analogous to previous one. For any $f \in L_2(0, 2T)$, w is uniquely solvable in a relevant class of functions in Δ^{2T}. So the above problem is a well-posed problem and its solution is determined by $q|_{0<x<cT}$.

Taking $f = \delta$ in the extended perturbed problem, one can contruct the fundamental solution

$$u^\delta = \delta(t - \tfrac{x}{c}) + w^\delta(x,t), \quad (x,t) \in \Delta^{2T},$$

and represent

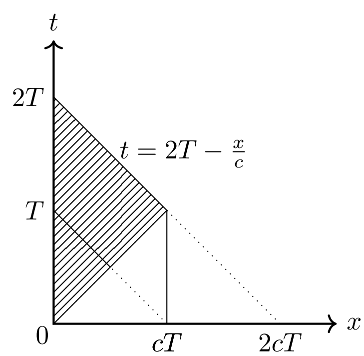
$$u^f(x,t) = f(t - \tfrac{x}{c}) + \int_0^{t-\frac{x}{c}} w^\delta(x, t-s) f(s) ds, \quad (x,t) \in \Delta^{2T},$$

that extends the domain of definition of the previous Duhamel representation.
Take a smooth control f provided $f(0) = 0$. Note when $x = 0$, $0 < t < 2T - \frac{0}{c} = 2T$. Dif and only iferentiating in above equation w.r.t. x by previous lemma, we get

$$\begin{aligned} u_x^f(0,t) &= -\frac{1}{c}f'(t-\frac{x}{c})\,\Big|_{x=0} - \frac{1}{c}w^\delta(x,t-(t-\frac{x}{c}))f(t-\frac{x}{c})ds\,\Big|_{x=0} + \int_0^{t-\frac{x}{c}} w_x^\delta(x,t-s)f(s)ds\,\Big|_{x=0} \\ &= -\sqrt{\rho}f'(t) - \sqrt{\rho}w^\delta(0,0)f(t-\frac{x}{c}) + \int_0^t w_x^\delta(0,t-s)f(s)ds \\ &= -\sqrt{\rho}f'(t) + \int_0^t r(t-s)f(s)ds, \quad 0 < t < 2T, \end{aligned}$$

where $r := w_x^\delta(0,\cdot)|_{0<t<2T}$ is called a *reply function*. It is a smooth function, which may have a jump at $t = 0$ only, and it will play a central role in the inverse problem.

Remark. By the locality principle, since w_x^δ for $0 < x < cT, 0 < t < 2T - \frac{x}{c}$ are determined by the values of the potential $q|_{0<x<cT}$ only, we have the reply function $r = w_x^\delta(0,\cdot)|_{0<t<2T}$ is determined by the part $q|_{0<x<cT}$ only.



2.3 String as dynamical system

2.3.1 System α^T

Here the original perturbed problem is considered in terms of the control theory and endowed with standard attributes of a dynamical system. The system is denoted by α^T.

Definition 2.1. The Hilbert space of controls (inputs) $\mathcal{F}^T : L_2(0,T)$ with the inner product

$$(f,g)_{\mathcal{F}^T} = \int_0^T f(t)g(t)dt$$

is said to be an *outer space* of the system α^T. It contains an increasing family of subspaces

$$F^{T,\xi} := \{f \in \mathcal{F}^T \mid f|_{t<T-\xi} = 0\}, \quad 0 \leqslant \xi \leqslant T$$

consisting of the delayed controls.

Definition 2.2. The space $\mathcal{H}^{cT} = L_{2,\rho}(\Omega^{cT})$ with the inner product

$$(u, v)_{\mathcal{H}^{cT}} = \int_{\Omega^{cT}} u(x)v(x)\rho dx$$

is called an *inner space* of the system α^T. For each $f \in \mathcal{F}^T$ and **each** $0 \leqslant t \leqslant T$, the wave $u^f(\cdot, t)$ is supported in Ω^{cT} by the property that $u^f|_{t<\frac{x}{c}} = 0, x \geqslant 0, t \geqslant 0$, and hence can be regarded as a time-dependent element of $\mathcal{H}^{cT}$. In control theory, $u^f(\cdot, t)$ is referred to as a state of the system at the moment t. So $\mathcal{H}^{cT}$ is a space of states.
The inner space contains an increasing family of subspaces

$$H^{c\xi} := \{y \in \mathcal{H}^{cT} \mid y|_{x>c\xi} = 0\}, \quad 0 \leqslant \xi \leqslant T.$$

Definition 2.3. Define a *control operator* $W^T : \mathcal{F}^T \to \mathcal{H}^{cT}$ in the system of α^T by sending the input f to the state $u^f(\cdot, T)$, which creates the waves, i.e.,

$$\begin{aligned}(W^T f)(x) = u^f(x, T) &= f(T - \frac{x}{c}) + \int_0^{T-\frac{x}{c}} w^\delta(x, T - s)f(s)ds \\ &= f(T - \frac{x}{c}) + \int_x^{cT} \frac{1}{c} w^\delta(x, \frac{s}{c})f(s)ds \\ &= f(T - \frac{x}{c}) + \int_x^{cT} w(x, s)f(s)ds, \quad x \in \Omega^{cT},\end{aligned}$$

where $w(x, s) := \frac{1}{c}w^\delta(x, \frac{s}{c})$ for any $x \leqslant s \leqslant cT$. As is easy to see, W^T is a bounded (continuous) operator.
Define the delay operator $D^{T,\xi} : \mathcal{F}^T \to \mathcal{F}^T$ given by $f \mapsto f_{T-\xi}$. By definition, $D^{T,\xi}\mathcal{F}^T = \mathcal{F}^{T,\xi}$. From the previous result, the delay of the control implies the delay of the wave and so

$$W^T D^{T,\xi} f = W^T f_{T-\xi} = u^{f_{T-\xi}}(\cdot, T) = u^f(\cdot, T - (T - \xi)) = u^f(\cdot, \xi) \in \mathcal{H}^{c\xi}.$$

Hence

$$W^T \mathcal{F}^{T,\xi} = W^T D^{T,\xi} \mathcal{F}^T \subseteq \mathcal{H}^{c\xi}, \quad 0 \leqslant \xi \leqslant T.$$

As we will see later, "$\subseteq$" can be replaced by "$=$".

Definition 2.4. Define a respond operator $R^T : \mathcal{F}^T \to \mathcal{F}^T$ in the system of α^T by sending the control f of the Sobolev class $H^1[0, T]$ with $f(0) = 0$, to the output $u^f_x(0, \cdot)$. Similarly, we have

$$(R^T f)(t) = u^f_x(0, t) = -\sqrt{\rho} f'(t) + \textstyle\int_0^t r(t - s)f(s)ds, \quad 0 < t < T.$$

In mechanics, $u^f_x(0, t)$ is interpreted as a value at time t of the **force** generated at the endpoint $x = 0$ of the string by the wave process initiated by the control f.
The presence of differentiation renders R^T to be an unbounded operator. Its adjoint $(R^T)^*$ is also unbounded. As is easy to show, it is defined on the controls f of the Sobolev class $H^1[0, T]$ with $f(T) = 0$ and acts by the rule

$$((R^T)^* f)(t) = \sqrt{\rho} f'(t) + \textstyle\int_t^T r(s - t)f(s)ds, \quad 0 < t < T.$$

Return to the extended problem and associate with it the extended reponse operator R^{2T} acting in the space $\mathcal{F}^{2T}$ on the controls f of the class $H^1[0,2T]$ with $f(0)=0$ by the rule

$$(R^{2T}f)(t) = u^f_x(x,t) = -\sqrt{\rho}f'(t) + \int_0^t r(t-s)f(s)ds, \quad 0<t<2T,$$

The extended response operator is one more intrinstic object of the system α^T. By the locality principle, R^{2T} is determined by the part $q|_{\Omega^{cT}}$.

Definition 2.5. Define a connecting operator $C^T : \mathcal{F}^T \to \mathcal{F}^T$ by letting $C^T = (W^T)^* W^T$. Since W^T is continuous, C^T is also continuous. By the definition, one has

$$(C^T f, g)_{\mathcal{F}^T} = \big((W^T)^* W^T f, g\big)_{\mathcal{F}^T} = (W^T f, W^T g)_{\mathcal{H}^{cT}} = (u^f(\cdot,T), u^g(\cdot,T))_{\mathcal{H}^{cT}},$$

i.e., C^T connects the metrics of the outer and inner spaces.
The connecting operator plays a key role in the BC-method owing to the remarkable fact: it can be represented via the response operator in explicit and simple form. To formulate the result we need to introduce auxiliary operators:

- The operator $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$,

  $$(S^T f)(t) := \begin{cases} f(t), & 0<t<T \\ -f(2T-t), & T \leqslant t < 2T \end{cases}$$

  extending the controls from $(0,T)$ to $(0,2T)$ by oddness w.r.t. $t=T$. As is easy to check, its adjoint $(S^T)^* : \mathcal{F}^{2T} \to \mathcal{F}^T$ acts by the rule

  $$\big((S^T)^* g\big)(t) = g(t) - g(2T-t) \quad 0<t<T.$$

- The integration operator $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$,

  $$(J^{2T} f)(t) = \int_0^t f(\eta)d\eta, \quad 0<t<2T$$

  The integration commutes with the response operator:

  $$R^{2T}J^{2T} = J^{2T}R^{2T}.$$

The formula, which express the connecting operator via the response operator, is

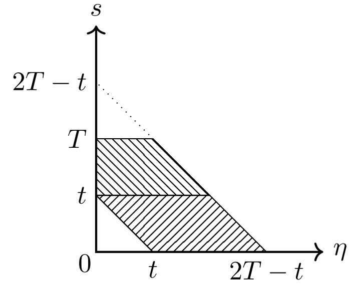
$$C^T = -\frac{1}{2}(S^T)^* R^{2T} J^{2T} S^T.$$

Then for $0 < t < T$, we have $f(s) = 0$ for any $T < s \leqslant 2T - t$ and so

$$
\begin{aligned}
(C^T f)(t) &= -\frac{1}{2}\left((S^T)^* R^{2T} J^{2T} S^T f\right)(t) = -\frac{1}{2}\left((S^T)^* R^{2T} J^{2T} f\right)(t) \\
&= -\frac{1}{2}\left((S^T)^* R^{2T}\right)\int_0^t f(s)ds = -\frac{1}{2}(S^T)^*\left(-\sqrt{\rho}f(t) + \int_0^t r(t-\eta)\left[\int_0^\eta f(s)ds\right]d\eta\right) \\
&= -\frac{1}{2}(S^T)^*\left(-\sqrt{\rho}f(t) + \int_0^t r(\eta)\left[\int_0^{t-\eta} f(s)ds\right]d\eta\right) \\
&= -\frac{1}{2}(S^T)^*\left(-\sqrt{\rho}f(t) + \int_0^t \left[\int_0^{t-s} r(\eta)d\eta\right]f(s)ds\right) \\
&= \frac{1}{2}\left(\sqrt{\rho}f(t) + \int_0^{2T-t}\left[\int_0^{2T-t-s} r(\eta)d\eta\right]f(s)ds - \int_0^t\left[\int_0^{t-s} r(\eta)d\eta\right]f(s)ds\right) \\
&= \frac{1}{2}\left(\sqrt{\rho}f(t) + \int_0^{T}\left[\int_0^{2T-t-s} r(\eta)d\eta\right]f(s)ds - \int_0^t\left[\int_0^{t-s} r(\eta)d\eta\right]f(s)ds\right) \\
&= \frac{1}{2}\sqrt{\rho}f(t) + \frac{1}{2}\iint_{\text{Shaded Area}} r(\eta)f(s)ds\eta \\
&= \frac{1}{2}\sqrt{\rho}f(t) + \frac{1}{2}\int_0^T\left[\mathbb{1}_{\{s\leqslant t\}}\int_{t-s}^{2T-t-s} r(\eta)d\eta + \mathbb{1}_{\{s>t\}}\int_0^{2T-t-s} r(\eta)d\eta\right]f(s)ds \\
&= \frac{1}{2}\sqrt{\rho}f(t) + \int_0^T \frac{1}{2}\left[\int_{t-s}^{2T-t-s} r(\eta)d\eta + \mathbb{1}_{\{s>t\}}\int_0^{2T-t-s} r(\eta)d\eta\right]f(s)ds \\
&= \frac{1}{2}\sqrt{\rho}f(t) + \int_0^T \frac{1}{2}\left[\int_{(t-s)^+}^{2T-t-s} r(\eta)d\eta\right]f(s)ds = \frac{1}{2}\sqrt{\rho}f(t) + \int_0^T c^T(t,s)f(s)ds,
\end{aligned}
$$

with the kernel

$$c^T(t,s) := \tfrac{1}{2}\int_{(t-s)^+}^{2T-t-s} r(\eta)d\eta, \quad 0 < s, t < T.$$



The following fact will be used later in solving the inverse problem. Assume that the external observer investigates the system α^T via its input-output correspondence. Such an observer operates at the endpoint $x = 0$ of the string; he can apply controls f and measure $u^f_x(0, \cdot)$ but, however, cannot see the waves u^f themselves on the string. **As result of such measurements**, the observer is provided with the reply function $r|_{(0,2T)} = w^\delta_x(0, \cdot)|_{(0,2T)}$. If so, the observer can determine the operator C^T be the representaion for $(C^T f)(t)$ above and then, for any given controls $f, g \in \mathcal{F}^T$, find the product of the waves $(u^f(\cdot,T),u^g(\cdot,T))_{\mathcal{H}^{cT}}$, even though the waves themselves are invisible! As we shall see, such an option enables one to make the waves visible.

2.3.2 Controllability

Can one manage the shape of the wave on a string? More precisely: Is it possible to drive the system of α^T from the initial zeor state to a given final state $u^f(\cdot,T)=y$ be means of the proper choice of the boundary control f? This sort of problems is a subject of the boundary control theroy, which is a highly developed branch of mathematical physics. Here the affirmative answer to the posed question is provided.

There is an evident necessary condition for the above-mentioned problem to be solvable. In view of the property $u^f|_{t<\frac{x}{c}}=0, x\geqslant 0, t\geqslant 0$, the function y has to be supported in the interval Ω^{cT} filled with waves at the final moment. Therefore, the relevant setup is: given function $y\in\mathcal{H}^{cT}$, to find control $f\in\mathcal{F}^T$ such that

$$u^f(x,T)=y(x),\quad x\in\Omega^{cT}$$

holds. It is what is called a *boundary control problem* (BCP). The BCP is obviously equivalent to the equation

$$W^Tf=y.$$

In the case of the unperturbed system, the BCP takes form of the equation

$$f(T-\tfrac{x}{c})=\tilde{u}^f(x,T)=y(x),\quad x\in\Omega^{cT},$$

which has the evident solution

$$f(t)=y(c(T-t)),\quad 0\leqslant t\leqslant T,$$

since $0\leqslant x\leqslant cT$.

In the perturbed case, letting

$$(W^Tf)(x)=f(T-\tfrac{x}{c})+\int_x^{cT}w(x,s)f(s)ds=y(x),\quad x\in\Omega^{cT},$$

then we have

$$f(t)+\int_{c(T-t)}^{cT}w(c(T-t),s)f(s)ds=y(c(T-t))\quad 0\leqslant t\leqslant T,$$

i.e.,

$$f(t)+\int_0^t cw(c(T-t),c(T-s))f(c(T-s))ds=y(c(T-t))\quad 0\leqslant t\leqslant T,$$

i.e.,

$$f(t)+\int_0^t k^T(t,s)f(c(T-s))ds=y(c(T-t))\quad 0\leqslant t\leqslant T,$$

with $k^T(t,s):=cw(c(T-t),c(T-s))$ that is a second-kind Volterra equation w.r.t. f. As is well-known, such an equatio and hence the BCP is unique solvable in $\mathcal{F}^T$ for any r.h.s.. In operator terms, this means that the control operator is boundly invertible, its inverse $(W^T)^{-1}$ being defined **onto** $\mathcal{H}^{cT}$.

This result can be also interpreted as follows. The set of waves

$$\mathcal{U}^T:=\{u^f(\cdot,T)\mid f\in\mathcal{F}^T\}=W^T\mathcal{F}^T$$

is called reachable at the moment T. The solvability of the BCP is equivalent to the relation

$$\mathcal{U}^T = \mathcal{H}^{cT}, \quad T \geqslant 0,$$

which shows that our system can be steered from zero state for any state by proper choice of the boundary control. In control theory, such a property of a dynamical system is referred to as a controllability. Taking into account the finiteness of the wave propagation speed, we specify it as a local boundary controllability of the system α^T.
For inverse problems, controllability is an affirmative and help property. A very general principle of system theory claim that the richer the set of states, which the observer can create in the system by means of the given reserve of controls, the richer information about the system, which the observer can extract from external measurements. As we shall see, the BC-method follows and realizes this principle.

2.3.3 Wave basis

Here we make use of controllability of the system α^T for storing up an efficient instrument, which will be used for solving the inverse problem. Fix $0 \leqslant \xi \leqslant T$.
Let $g \in \mathcal{H}^{c\xi} \subseteq \mathcal{H}^{cT}$. Then $g|_{x>c\xi} = 0$. Since W^T is invertible, there exists $h \in \mathcal{F}^T$ such that $W^T h = u^h(\cdot, T)$ with $W^T h|_{0<x<\xi} = g = u^h(\cdot, \xi) = W^T D^{T,\xi} h \in W^T D^{T,\xi} \mathcal{F}^T$. So

$$W^T \mathcal{F}^{T,\xi} = W^T D^{T,\xi} \mathcal{F}^T = \mathcal{H}^{c\xi}, \quad 0 \leqslant \xi \leqslant T.$$

Since $W^T : \mathcal{F}^T \to \mathcal{H}^{cT}$ is invertible, $W^T|_{\mathcal{F}^{T,\xi}} : \mathcal{F}^{T,\xi} \to \mathcal{H}^{c\xi}$ is also invertible.
Let us choose a basis of controls $\{f_j^\xi\}_{j=1}^\infty$ in the subspace $\mathcal{F}^{T,\xi}$. Since W^T is a boundly invertible operator, the above reltion yields that the corresponding waves $\{u^{f_j^\xi}(\cdot, T)\}_{j=1}^\infty$ constitute a basis of the subspace $\mathcal{H}^{c\xi}$. For the needs of the inverse problem, it is convenient to make the **latter** basis orthonormalized. To this end, we recast the basis of controls by the Schmidt orthogonalization process w.r.t. the bilinear form $(C^T f, g)_{\mathcal{F}^T}$:

$$\begin{array}{ll}
\tilde{g}_1^\xi = f_1^\xi, & g_1^\xi = \frac{\tilde{g}_1^\xi}{\sqrt{\left(C^T \tilde{g}_1^\xi, \tilde{g}_1^\xi\right)_{\mathcal{F}^T}}}, \\
\tilde{g}_2^\xi = f_2^\xi - (C^T f_2^\xi, g_1^\xi)_{\mathcal{F}^T} g_1^\xi, & g_2^\xi = \frac{\tilde{g}_2^\xi}{\sqrt{\left(C^T \tilde{g}_2^\xi, \tilde{g}_2^\xi\right)_{\mathcal{F}^T}}}, \\
\vdots & \vdots \\
\tilde{g}_k^\xi = f_k^\xi - \sum_{j=1}^{k-1} (C^T f_k^\xi, g_j^\xi)_{\mathcal{F}^T} g_j^\xi, & g_k^\xi = \frac{\tilde{g}_k^\xi}{\sqrt{\left(C^T \tilde{g}_k^\xi, \tilde{g}_k^\xi\right)_{\mathcal{F}^T}}}, \\
\vdots & \vdots
\end{array}$$

and get a new system of controls $\{g_k^\xi\}_{k=1}^\infty$, which is also a basis in $\mathcal{F}^{T,\xi}$ and satisfies

$$\left(C^T g_k^\xi, g_l^\xi\right)_{\mathcal{F}^T} = \delta_{kl}, \quad k, l = 1, 2, \cdots$$

by contruction. The corresponding waves $\{u_k^\xi\}_{k=1}^\infty$, where $u_k^\xi := u^{g_k^\xi}(\cdot, T) = W^T g_k^\xi$ form a basis in the subspace $\mathcal{H}^{c\xi}$, the basis turning out to be **orthonormal**. Indeed, we have

$$\left(u_k^\xi, u_l^\xi\right)_{\mathcal{H}^{c\xi}} = \left(u^{g_k^\xi}(\cdot, T), u_l^{g_l^\xi}(\cdot, T)\right)_{\mathcal{H}^{c\xi}} = (C^T g_k^\xi, g_l^\xi)_{\mathcal{F}^{T,\xi}} = \delta_{kl}, \quad k, l = 1, 2, \cdots .$$

We say $\{u_k^\xi\}_{k=1}^\infty$ to be a wave basis in $\mathcal{H}^{c\xi}$.

2.4 Truncation

As in the previous subsection, we keep $0 \leqslant \xi \leqslant T$ fixed. In the inner space $\mathcal{H}^{cT}$ introduce the operation $P^{c\xi}$. Let $y \in \mathcal{H}^{cT}$, define

$$(P^{c\xi}y)(x) := \begin{cases} y(x), & x \in \Omega^{c\xi} \\ 0, & x \in \Omega^{cT} \setminus \Omega^{c\xi} \end{cases}$$

that truncates functions onto the subinterval $\Omega^{c\xi} \subseteq \Omega^{cT}$. As is easy to see, $P^{c\xi}$ is the orthogonal projection in $\mathcal{H}^{cT}$ onto the subspace $\mathcal{H}^{c\xi}$, and expanding over the wave basis, we can represent the truncated function in the form of the Fourier series:

$$P^{c\xi}y = y|_{x\in\Omega^{c\xi}} = \sum_{k=1}^{\infty} \left(y, u_k^\xi\right)_{\mathcal{H}^{cT}} u_k^\xi.$$

By the controllability of α^T, the function $y \in \mathcal{H}^{cT}$ can be regarded as a wave produced by a control $f = (W^T)^{-1}y$. By the same reason and since $W^T|_{\mathcal{F}^{T,\xi}} : \mathcal{F}^{T,\xi} \to \mathcal{H}^{c\xi}$ is invertible, the truncated function $P^{c\xi}y \in \mathcal{H}^{c\xi}$ is also a wave produced by a control $f^\xi := (W^T)^{-1}P^{c\xi}y \in \mathcal{F}^{T,\xi}$. Thus, we have a correspondence $f \mapsto f^\xi$, which is realized by an operator

$$P^\xi := (W^T)^{-1}P^{c\xi}W^T$$

acting in the outer space $\mathcal{F}^T$. In other words, the truncating $y \mapsto P^{c\xi}y$ in the inner space induces a truncation-like operation $f \mapsto P^\xi f$ in the outer space.

$$\begin{array}{ccc} \mathcal{F}^T & \xrightarrow{W^T} & \mathcal{H}^{cT} \\ \downarrow{\scriptstyle P^\xi} & & \downarrow{\scriptstyle P^{c\xi}} \\ \mathcal{F}^{T,\xi} & \xrightarrow{W^T} & \mathcal{H}^{c\xi} \end{array}$$

An important fact is that the latter operation can be represented in the form of the expansion over the system $\{g_k^\xi\}_{k=1}^\infty$: namely, let $f \in \mathcal{F}^T$, then

$$P^{c\xi}W^Tf = \sum_{k=1}^{\infty} \left(W^Tf, u_k^\xi\right)_{\mathcal{H}^{cT}} W^Tg_k^\xi = \sum_{k=1}^{\infty} \left(W^Tf, W^Tg_k^\xi\right)_{\mathcal{H}^{cT}} W^Tg_k^\xi = W^T\sum_{k=1}^{\infty} \left(C^Tf, g_k^\xi\right)_{\mathcal{F}^T} g_k^\xi,$$

i.e.,

$$P^\xi f = (W^T)^{-1}P^{c\xi}W^Tf = \sum_{k=1}^{\infty} \left(C^Tf, g_k^\xi\right)_{\mathcal{F}^T} g_k^\xi.$$

2.4.1 Amplitude formula

Now, let us derive a relation, which represents the values of the waves through the operator P^ξ. Let a control $f \in \mathcal{F}^T$ be **continuous** on $[0, T]$. Since W^T is bounded, by the definition of the continuity, we have $u^f(\cdot,T)$ is also continuous on Ω^{cT}. Also, since $P^{c\xi}$ is a truncation operation, $(P^{c\xi}u^f(\cdot,T))(x)$ vanishes for $x > c\xi$, is continuous in $\Omega^{c\xi}$ and has a jump

$$\left(P^{c\xi}u^f(\cdot,T)\right)(x)\,\Big|_{x=c\xi-0}^{x=c\xi+0} = 0 - u^f(c\xi,T) = -u^f(c\xi,T).$$

Also, since $P^{c\xi}u^f(\cdot,T) = P^{c\xi}W^Tf = W^TP^\xi f = u^{P^\xi f}(\cdot,T)$ and $P^\xi f \in \mathcal{F}^{T,\xi}$, we have

$$\left(P^{c\xi}u^f(\cdot,T)\right)(x)\,\Big|_{x=c\xi-0}^{x=c\xi+0} = \left(u^{P^\xi f}(\cdot,T)\right)(x)\,\Big|_{x=c\xi-0}^{x=c\xi+0} = -(P^\xi f)(s)\,\Big|_{s=T-\xi-0}^{s=T-\xi+0} = -(P^\xi f)(T-\xi+0).$$

Compare the 2 equations above, we have

$$u^f(c\xi,T) = (P^\xi f)(T-\xi+0) = \left[\sum_{k=1}^{\infty}\left(C^Tf, g_k^\xi\right)_{\mathcal{F}^T} g_k^\xi(t)\right]\Bigg|_{t=T-\xi+0},$$

which is the *amplitude formula* (AF). The reason is that it represents the wave through the jump amplitudes of the control, which appears as one projects the control on the subspaces $\mathcal{F}^{T,\xi}$ by the operators P^ξ. The backgroup of the AF is geometrical optics; its various versions play the role of key tool of the BC-method.

2.4.2 Special BCP

In this subsection we deal with the controlling problem of $u^f(x,T) = y(x), x \in \Omega^{cT}$ with a special r.h.s. $y \in \mathcal{H}^{cT}$. Consider the Cauchy problem

$$-p'' + q(x)p = 0,\ \ x > 0,$$
$$p|_{x=0} = \alpha,\ \ p'|_{x=0} = \beta,$$

where α and β are constant. By standard ODE theory, such a problem has a unique **smooth** solution $p = p_{\alpha\beta}(x)$. In what follows we assume α and β are fixed. Therefore we omit these subsripts and write just $p(x)$.
Consider the special BCP: find a pieice-wise continuous control $f \in \mathcal{F}^T$ satisfying

$$u^f(\cdot,T) = p, \quad x \in \Omega^{cT}.$$

By previous result, such a problem has a unique solution $f^T \in \mathcal{F}^T$ and applying the AF for $f = f^T$ we get

$$p(c\xi) = u^f(c\xi,T) = (P^\xi f)(T-\xi+0) = \left[\sum_{k=1}^{\infty}\left(C^Tf, g_k^\xi\right)_{\mathcal{F}^T} g_k^\xi(t)\right]\Bigg|_{t=T-\xi+0}, \quad 0 < \xi \leqslant T.$$

The remarkable fact is that, in this case, the coefficients of the series can be found explicitly.
To find the coefficients, let us take a smooth control $g \in \mathcal{F}^T$ provided $g(0) = g'(0) = g''(0) = 0$. The corresponding wave u^g satisfies the equations in the perturbed problem in the classic sense; therefore it vanishes at its forward front together with the derivatives and we have the boundary condition

$$u^g(cT,t) = 0 = u^g_x(cT,t), \quad 0 \leqslant t \leqslant T.$$

This enables one to justify the following calculations:

$$
\begin{aligned}
(C^T f, g)_{\mathcal{F}^T} &= \left(u^f(\cdot,T), u^g(\cdot,T)\right)_{\mathcal{H}^{cT}} = (p, u^g(\cdot,T))_{\mathcal{H}^{cT}} = \int_0^{cT} p(x)u^g(x,T)\rho dx \\
&= \int_0^{cT} p(x)u^g(x,t)\Big|_{r=0}^{r=T} \rho dx = \int_0^{cT} p(x)\left[\int_0^T u_t^g(x,r)dr\right]\rho dx \\
&= \int_0^{cT} p(x)\left[\int_0^T u_t^g(x,s)\Big|_{s=0}^{s=r} dr\right]\rho dx = \int_0^{cT} p(x)\left[\int_0^T \left[\int_0^r u_{tt}^g(x,s)ds\right]dr\right]\rho dx \\
&= \int_0^{cT} p(x)\left[\int_0^T u_{tt}^g(x,s)\left[\int_s^T dr\right]ds\right]\rho dx = \int_0^{cT} p(x)\left[\int_0^T (T-s)u_{tt}^g(x,s)dt\right]\rho dx \\
&= \int_0^T (T-s)\left[\int_0^{cT} p(x)\rho u_{tt}^g(x,s)dx\right]ds \\
&= \int_0^T (T-s)\left[\int_0^{cT} p(x)\left[u_{xx}^g(x,s) - q(x)u^g(x,s)\right]dx\right]ds \\
&= \int_0^T (T-s)\left[\int_0^{cT} p(x)du_x^g(x,s) - \int_0^{cT} q(x)p(x)u^g(x,s)dx\right]ds \\
&= \int_0^T (T-s)\left[u_x^g(x,s)p(x)\Big|_{x=0}^{x=cT} - \int_0^{cT} p'(x)u_x^g(x,s)dx - \int_0^{cT} q(x)p(x)u^g(x,s)dx\right]ds \\
&= \int_0^T (T-s)\left[-u_x^g(0,s)p(0) - \int_0^{cT} p'(x)du^g(x,s) - \int_0^{cT} q(x)p(x)u^g(x,s)dx\right]ds \\
&= \int_0^T (T-s)\Bigg[-u_x^g(0,s)p(0) - u^g(x,s)p'(x)\Big|_{x=0}^{x=cT} + \int_0^{cT} p''(x)u^g(x,s)dx \\
&\quad - \int_0^{cT} q(x)p(x)u^g(x,s)dx\Bigg]ds \\
&= \int_0^T (T-s)\left[-p(0)u_x^g(0,s) + p'(0)u^g(0,s) + \int_0^{cT}\left[p''(x) - q(x)p(x)\right]u^g(x,s)dx\right]ds \\
&= \int_0^T (T-s)\left[-\alpha(R^T g)(s) + \beta g\right]ds = \left(\varkappa, -\alpha R^T g + \beta g\right)_{\mathcal{F}^T} = \left(-(R^T)^*\varkappa\alpha + \varkappa\beta, g\right)_{\mathcal{F}^T}
\end{aligned}
$$

where $\varkappa(t) := T - t$ for any $0 \leqslant t \leqslant T$ and $\varkappa \in \mathcal{F}^T$. Thus, we have got the equality

$$
(C^T f, g)_{\mathcal{F}^T} = \left(-(R^T)^*\varkappa\alpha + \varkappa\beta, g\right)_{\mathcal{F}^T}.
$$

For each $k \in \mathbb{N}$, letting $g = g_k^\xi$, we have

$$
(C^T f, g_k^\xi)_{\mathcal{F}^T} = \left(-(R^T)^*\varkappa\alpha + \varkappa\beta, g_k^\xi\right)_{\mathcal{F}^T}.
$$

Thus,

$$p(c\xi) = u^f(c\xi, T) = \left[\sum_{k=1}^{\infty} \left(-(R^T)^* \varkappa\alpha + \varkappa\beta, g_k^\xi\right)_{\mathcal{F}^T} g_k^\xi(t)\right]\Bigg|_{t=T-\xi+0}, 0 < \xi \leqslant T.$$

The significance of this formula is that it represents the function p, which is an object of the inner space, through the objects of the outer space. We shall see that it is the fact, which enables the external observer to use the above expression for solving the inverse problem.

2.5 Inverse Problem

2.5.1 Statement

The setup of the dynamical inverse problem is motivated by the locality principle or, more exactly, by the local character of dependence of the response operator on the potential. Recall that the operator R^{2T} associated with the extended perturbed problem is determined by $q|_{\Omega^{cT}}$. This property is of transparent physical meaning. Namely, the response of the system α^T (i.e., the force $u_x^f(0,t)$) measured at the endpoint on the string) on the action of a control f is formed by the waves, which are reflected from inhomogeneities of the string and return back to the endpoint $x = 0$. Since the wave propagation speed is equal to c, the waves reflected from the depths $x > cT$ return to the endpoint later than $t = 2T$ and then they are not recorded by the external observer measuring the values $u_x^f(0,t)$ for times $0 \leqslant t \leqslant 2T$. Therefore, the operator R^{2T}, which corresponds to these measurements, contains certain information on $q|_{\Omega^{cT}}$ but "knows nothing" about $q|_{x>cT}$. Taking into account this fact, the relevant statement of the dynamical inverse problem has to be as follows: **given operator R^{2T}, recover the potential $q|_{\Omega^{cT}}$.**

Note in addition that this statement agrees closely with very general principles of the system theory. By one of them, the input-output map of a linear system (here the response operator R^{2T}) determines not the whole system but its controllable part (here the interval Ω^{cT}) only.

2.5.2 Solving inverse problem

The potential $q|_{\Omega^{cT}}$ can be recovered by means of the following procedure:

(a) Determine the constant $\sqrt{\rho}$ and the reply function $r|_{(0,2T)}$. Find the operator C^T and the compute $(R^T)^*$.

(b) Fix $\xi \in (0,T]$. In the subspace $\mathcal{F}^{T,\xi} \subseteq \mathcal{F}^T$, construct a C^T-orthogonal basis of controls $\{g_k^k\}_{k=1}^{\infty}$.

(c) Choose two constant α, β provided $\alpha^2 + \beta^2 \neq 0$ and find the value $p(c\xi)$.

(d) Varying $\xi \in (0,T]$, recover the functon p in Ω^{cT}.

(e) Determine the potential by

$$q(x) = \frac{p''(x)}{p(x)}, \quad x \in \Omega^{cT}.$$

Using the step (e), one has to take care of the case $p(x) = 0$. However, as is well-known, on any finite interval Ω^{cT}, there may be only a finite number of zeros of p. Therefore, if q is determined in other points it can be extended to the zeros by continuity. Another option to remove the zero of p at a given $x = x_0$ is to choose another α, β.

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