1 Simplicial complexes and Face ideals

Let G be a graph with vertex set $V = \{v_1, \dots, v_d\}$.

Remark. The previous section gave a method for computing m-irreducible decompositions for quadratic square-free monomial ideals. The next section introduces some tools to accomplish this for arbitrary square-free monomial ideals. This uses the notation of a simplicial complex. One often thinks of this as a higher dimensional graph, not only does it have vertices and edges, but it also can have shaded triangles, solid tetrahedra.

Definition 1.1. A *simplicial complex* V is a non-empty collection Δ of subsets of V that is closed under subsets, that is, if $F \subseteq G \subseteq V$ and $G \in \Delta$, then $F \in \Delta$.

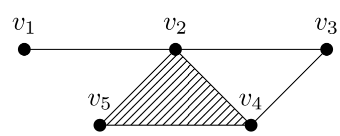
(a) An element of Δ is called a *face*.

(b) An element of the form $\{v_i\}$ is called a *vertex* of Δ.

(c) An element of the form $\{v_j, v_k\}$ is called an *edge* of Δ.

(d) A maximal element of Δ with respect containment is a *facet* of Δ.

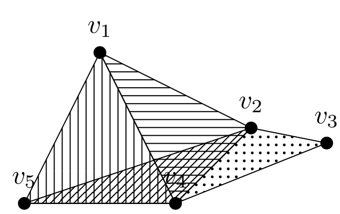
Example 1.2. Consider



This is the simplicial complex with the following faces:

- trivial: $\emptyset$.
- vertices: $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{4_5\}$.
- edges: $\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\}$.
- shaded triangle: $\{v_2, v_4, v_5\}$.
- facets: $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_4, v_5\}$.

Example 1.3. Consider



This is the simplicial complex with the following faces:

- trivial: $\emptyset$.

- vertices: $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{4_5\}$.
- edges: $\{v_1, v_2\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\}$.
- shaded triangle: $\{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_4, v_5\}$.
- solid tetrahedron: $\{v_1, v_2, v_4, v_5\}$.
- facets: $\{v_2, v_3, v_4\}, \{v_1, v_2, v_4, v_5\}$.

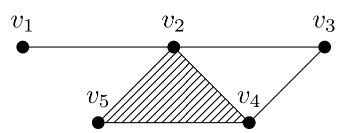
Let Δ be a simplicial complex on V.

Definition 1.4. The *face ideal* of R associated to Δ is the ideal generated by the non-faces of Δ:

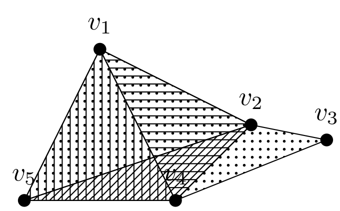
$$J_\Delta = (X_{i_1} \cdots X_{i_s} \mid 1 \leqslant i_1 < \cdots < i_s \leqslant d \text{ and } \{v_{i_1}, \ldots, v_{i_s}\} \notin \Delta).$$

Example 1.5.

$$J_\Delta = (X_1X_3, X_1X_4, X_1X_5, X_2X_3X_4) = (X_1, X_3) \cap (X_1, X_2, X_5) \cap (X_1, X_4, X_5) \cap (X_3, X_4, X_5).$$



Example 1.6. $J_\Delta = (X_1X_3, X_3X_5)R = (X_3) \cap (X_1, X_5)$.



Definition 1.7. $F \subseteq V$ is *independent* in G if none of the vertices in F are adjoint in G.

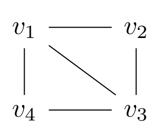
An independent set is *maximal* if it is maximal respect to containment.

Let Δ_G denote the set of independent subsets of G. This is the *independence complex* of G.

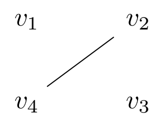
Remark. Every singleton $\{v_i\} \subseteq V$ is independent in H, as is the empty set $\emptyset \subseteq V$.

Since every subset of an independent set in G is also independent in G, Δ_G is a simplicial complex on V.

Example 1.8. Consider



Then $\Delta_G = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_2, v_4\}\}$. That is, the geometric realization of Δ is as follows:



The maximal independent subsets in G are $\{v_1\}, \{v_3\}$ and $\{v_2, v_4\}$.

Remark. The next result shows that the faces of Δ_G are in bijection with vertex covers of G, and the facets of Δ_G are in bijection with the minimal vertex covers of G.

Lemma 1.9. (a) $F \subseteq V$ is independent if and only if $V \smallsetminus F$ is a vertex cover of G.

(b) An independent $F \subseteq V$ is maximal if and only if $V \smallsetminus F$ is minimum as a vertex cover.

Remark. A minimal vertex cover is complementary to a maximal independent set.

Theorem 1.10. $I(G) = J_{\Delta_G}$.

Proof. "$\subseteq$". Let X_iX_j be a generator of $I(G)$ given by the edge $v_iv_j \in E$. Then $\{v_i, v_j\}$ is not independent in G. So $X_iX_j \in J_{\Delta_G}$.

"$\supseteq$". Let $X_{i_1} \cdots X_{i_n}$ be one of the generators of J_{Δ_G} given by the non-face $\{v_{i_1}, \ldots, v_{i_n}\} \notin \Delta_G$. Then $\{v_{i_1}, \ldots, v_{i_n}\}$ is not independent in G. So it must contain a pair of adjacent vertices v_{i_k}, v_{i_m}. It follows that $X_{i_k}X_{i_m}$ is a generator of I_G. Thus, $X_{i_1} \cdots X_{i_n} \in (X_{i_k}X_{i_m})R \subseteq I(G)$. $\square$

Example 1.11. Continue on previous example and note the non-faces of Δ_G are

$$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\},$$

$$\{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_4\}.$$

After removing redundancies, we see

$$J_{\Delta_G} = (X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_3X_4)R = I_G.$$

Definition 1.12. The *dimension* of a face $F \in \Delta$ is $|F| - 1$. The *dimension* of Δ, denoted $\dim(\Delta)$ is the maximal dimension of a face Δ.

The simplicial complex Δ is pure if all facets of Δ have the same dimension.

For $i = -1, \ldots, \dim(\Delta)$, let

$$f_i(\Delta) := \text{ the number of elements of } \Delta \text{ of dimension } i.$$

The f-vector of Δ is the vector

$$f(\Delta) := (f_0(\Delta), f_1(\Delta), \ldots, f_{\dim(\Delta)}(\Delta)).$$

2 Decomposition of face ideals

Let Δ be a simplicial complex on V.

Definition 2.1. Let $F \subseteq V$, define

$$Q_F = P_{V \smallsetminus F} = \langle X_i \mid v_i \notin F \rangle.$$

Example 2.2. Let $d = 5$, then $Q_{\{v_2,v_3,v_4\}} = \langle x_1, x_5\rangle$, $Q_V = \langle\emptyset\rangle = 0$ and $Q_\emptyset = \mathfrak{X} := \langle X_1, \dots, X_5\rangle$.

Remark (Facts)**.** (a) Let $F_1, F_2 \subseteq V$, then $Q_{F_1} \subseteq Q_{F_2}$ if and only if $F_2 \subseteq F_1$.

(b) $J \lneq_m R$ is square-free if and only if there exist $F_1, \dots, F_n \subseteq V$ such that $J = \bigcap_{i=1}^n Q_{F_i}$.

Lemma 2.3. Let $F \subseteq V$, then $J_\Delta \subseteq Q_F$ if and only if $F \in \Delta$.

Proof. Let $F = \{v_{i_1}, \dots, v_{i_n}\} \subseteq V$ and $V \smallsetminus F = \{v_{j_1}, \dots, v_{j_p}\}$.

$\Longrightarrow$ Assume $J_\Delta \subseteq Q_F$. Suppose $F \notin \Delta$, then by definition $X_{i_1} \cdots X_{i_n} \in J_\Delta \subseteq Q_F = \langle X_{j_1}, \dots, X_{j_p}\rangle$. So there exists $q \in \{1, \dots, p\}$ such that $X_{j_q} \mid X_{i_1} \cdots X_{i_n}$. Then there exists $l \in \{1, \dots, n\}$ such that $j_q = i_l$, a contradiction.

$\Longleftarrow$ Assume $F \in \Delta$. Let $X_{r_1} \cdots X_{r_q} \in J_\Delta$ be a generator. Then $V' = \{v_{r_1}, \dots, v_{r_q}\} \notin \Delta$. Since $F \in \Delta$, $V' \nsubseteq F$. So there exists $s \in \{1, \dots, q\}$ such that $v_{r_s} \in V' \smallsetminus F$, i.e., $X_{r_s} \in Q_F$. Thus, $X_{r_1} \cdots X_{r_q} \in (X_{r_s}) \subseteq Q_F$. □

Theorem 2.4. $J_\Delta = \bigcap_{F\in\Delta} Q_F = \bigcap_{F \textit{ facet of } \Delta} Q_F$.

Proof. Both "⊆⊆" are trivial. Since every face $F \in \Delta$ is contained in a facet F', $Q_F \supseteq Q_{F'}$. Removing redundancies from $\bigcap_{F\in\Delta} \Delta Q_F$, we get $\bigcap_{F \text{ facet of } \Delta} Q_F$. Next, since J_Δ is square-free, there are $F_1, \dots, F_n \subseteq V$ such that $J_\Delta = \bigcap_{i=1}^n Q_{F_i} \subseteq Q_{F_i}$ for $i = 1, \dots, n$. By previous lemma, $F_i \in \Delta$ for $i = 1, \dots, n$. So $\{F_1, \dots, F_n\} \subseteq \Delta$. Thus, $\bigcap_{F\in\Delta} Q_F \subseteq \bigcap_{i=1}^n Q_{F_i} = J_\Delta$. □

Lemma 2.5. Let $V' \subseteq V$. Then $I_G \subseteq Q_{V'}$ if and only if V' is independent.

Proof. $\Longrightarrow$ Assume $I_G \subseteq Q_{V'}$. Suppose there exist $v_i, v_j \in V'$ such that $v_i \sim v_j$. Then we have $X_iX_j \in I(G) \subseteq Q_{V'} = \langle X_k \mid v_k \notin V'\rangle$. So $X_k \mid X_iX_j$ for some $v_k \notin V'$. Hence $k = i$ or $k = j$. Thus, $V' \not\ni v_k = v_i \in V'$ or $V' \not\ni v_k = v_j \in V'$, a contradiction.

$\Longrightarrow$ Assume V' is independent. Let X_iX_j be a generator of I_G for some $i, j \in \{1, \dots, d\}$. Then $v_i \sim v_j$. So $v_i \notin V'$ or $v_j \notin V'$. Then $v_i \in V \smallsetminus V'$ or $v_j \in V \smallsetminus V'$. So $X_i \in Q_{V'}$ or $X_j \in Q_{V'}$. Thus, $X_iX_j \in Q_{V'}$. □

Theorem 2.6. $I(G) = \bigcap_{V' \textit{ indep.}} Q_{V'} = \bigcap_{V' \textit{ max. independ.}} Q_{V'}$. *These are m-irreducible decompositions and the second one is irredundant.*

Proof. It follows from the $I(G) = J_{\Delta_G}$ the definition of Δ_G. □

3 Facet ideal and their decompositions

Let Δ be a simplicial complex on V.

Definition 3.1. The *facet ideal* of R associated to Δ is the ideal generated by the facets of Δ:

$$K_\Delta = (X_{i_1} \cdots X_{i_s} : 1 \leqslant i_1 < \cdots < i_s \leqslant d \text{ and } \{v_{i_1}, \dots, v_{i_n}\} \text{ is a facet in } \Delta).$$

Remark. The facet ideal K_Δ is square-free. Moreover, since the facets of Δ are incomparable with respect to containment, they generate K_Δ irredundantly.

Definition 3.2. A *vertex cover* of Δ is a subset $V' \subseteq V$ such that for any facet $F \in \Delta$, $F \cap V' \neq \emptyset$.

A vertex cover of Δ is *minimal* if it does not properly contain another vertex cover of Δ.

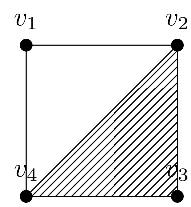
Lemma 3.3. Let $V' \subseteq V$. Then $K_\Delta \subseteq P_{V'}$ if and only if V' is a vertex cover of Δ.

Proof. As for edge ideals. $\square$

Theorem 3.4. $K_\Delta = \bigcap_{V' \text{ v.cover of } \Delta} P_{V'} = \bigcap_{V' \text{ min. v.cover of } \Delta} P_{V'}$. *These are m-irreducible decompositions and the second one is irredundant.*

Proof. As for edge ideal and use previous theorem. $\square$

Example 3.5. Let $J = \langle X_1X_2, X_2X_3X_4, X_1X_4\rangle$. Find a simplicial complex Δ on V such that $J = K_\Delta$. The geometric realization of Δ is the following.



Next, we list the minimal vertex cover of Δ: $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_4\}$. Then by previous theorem, $J = K_\Delta = \langle X_1, X_2\rangle \cap \langle X_1, X_3\rangle \cap \langle X_1, X_4\rangle \cap \langle X_2, X_4\rangle$.